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ÚSTAV MATEMATIKY

THE PROBLEM OF ENERGY-EFFICIENT TRAIN CONTROL
PROBLÉM ENERGETICKY OPTIMÁLNÍ JÍZDY VLAKU

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Specification Master's Thesis

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Supervisor:	**prof. RNDr. Jan Čermák, CSc.**
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

The problem of energy–efficient train control

Concise characteristic of the task:

The dynamics of an electric train journey is usually studied with respect to a search for its energy–efficient control. This problem can be solved in the frame of different variants by use of suitable methods of continuous and discrete optimization.

Goals Master's Thesis:

A survey of basic mathematical models used in energy–efficient train controls.
Application of Pontryagin's Maximum Principle to a given model.
Analysis of optimal driving regimes.
A discussion of energy–efficient train timetabling.

Recommended bibliography:

HOWLETT, P. G., PUDNEY, P. J. Energy-Efficient Train Control. Springer, London, UK 1995.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2018/19

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prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Abstract

The Diploma thesis deals with the problem of energy-efficient train control. It presents the basic survey of mathematical models used in the problem of energy-efficient train control, analysis of optimal driving regimes, determining optimal switching times between optimal driving regimes and timetabling of the train.

The mathematical formulation of the problem is done using Newton's second law of motion and other known physical laws. To analyse optimal driving regimes and determine the switching times between optimal driving regimes, we apply tools of optimal control theory, particularly Pontryagin's Maximum Principle. The timetabling of the train is discussed from the numerical solution of the settled non-linear programming problem.

KEYWORDS

Energy-efficient train control, Optimal control, Optimal driving regimes, Switching times, Pontryagin's Maximum Principle

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I declare that I have worked on this thesis independently under a supervision of prof. RNDr. JanČermák, CSc.and using the sources listed in the bibliography.

Zewude Alemayehu Berkessa

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This thesis is dedicated to my Son, Boru who is one year old.

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1 Introduction

1.1 A Short Story of the Problem

The problem of an energy-efficient train control was typically started in 1980 as an active research topic [14]. In this year, Iran Milroy did his Ph.D. dissertation entitled "Aspects of Automatic Train Control". In his work, he showed that for short journeys an energy-efficient driving strategy has three control phases. These are maximum acceleration, coast, and maximum braking. In 1982, he formed the Transport Control Group at the South Australian Institute of Technology to work on a project funded by the South Australian Department of Transport. The aim was to determine whether the suggested driving strategies were effective in practice and if so, to develop a system for achieving fuel saving on suburban trains in Adelaide. The first part of the project involved calculating efficient speed profiles for various section of track on the Adelaide rail network. As a result, they found on each trip the train completed the section within a few seconds of the desired time and the time spent accelerating was much less than normal. Then, the project continued to determine and build a system that could compute an efficient driving strategy in real time and display appropriate driving advice to the driver. The algorithm and software for the system were developed by Basil Benjamin and a post-graduate Computer Studies student, Peter Pudney. The computer hardware was designed and built by Tony Gelonese from the School of Electrical Engineering and the resulting system known as *Metromiser*, which monitored the state of a journey using stored timetable and route information, advised the driver when to coast and brake so that the train arrived at each stop on time and consumed as minimum energy as possible. Metromiser was first evaluated in Adelaide in February 1985 and the unit was fitted to a diesel-hydraulic railcar running in normal service on an 80 minutes round trip. Twenty trips were evaluated: ten trips with driving advice and ten trips without advice. Metromiser has achieved a fuel saving of 15% and significant improvements in timekeeping [2].

1.2 Statement of the Problem

The problem of energy-efficient train control is usually described as minimizing the energy consumption of a train traveling from one station to the next within a given time period. It belongs to the optimal control theory that can be solved with the use of Pontryagin's Maximum Principle (PMP) to find optimal driving regimes that make up the optimal energy-efficient driving strategy of a train under different conditions. The sequence of optimal driving regimes and switching times between the optimal driving regimes are the main change of the problem since they are not trivial in general. This leads to a wide range of nonlinear programming and algorithms to compute optimal train trajectories.

The optimal control theory is a powerful tool that gives the ability to deal with complex control problems. It requires an advanced mathematical and dynamic programming background and is already famous for its adaptability and quality of results. [20] states that optimal control theory is a mature mathematical discipline with numerous applications in both science and engineering. Energy-efficient train control is an effective mean to reduce operating energy costs and so a lot of research is devoted to this area. Most of this research is based on optimal control theory, and in particular Pontryagin's Maximum Principle (PMP) has been applied to derive the necessary optimality conditions that characterize the optimal driving regimes [12].

This thesis deals with the problem of energy-efficient train control by considering the following cases:

- Minimum-time problem.
- Minimum-energy problem.
- Minimum-time-energy problem.

1.3 The Objectives and Structure of the Thesis

The objectives of the thesis are:

- To review a basic mathematical models used in energy-efficient train controls.

- To apply Pontryagin's Maximum Principle to a given model.
- To analysis of optimal driving regimes.
- To discussion energy-efficient train timetabling.

The significance of the thesis is:

- Result of the thesis expected to find optimal driving modes that a train is driven with least amount of energy consumption.
- A lot of money can be saved by decreasing the energy consumption of the trains.
- Besides, there are external benefits like decrease in Co_2 emission.
- Further more, the result of this thesis can also be used as a source of information and invites other researchers for further studies of the problem of energy-efficient train control.

The structure of this thesis is following: In section 2, we give a survey of a basic mathematical models used in the problem of energy-efficient train control. These are: electric energy, and fuel consumption models. In section 3, we give a brief explanation on the basic optimal control theory and Pontryagin's Maximum Principle (PMP). In section 4, we give the main formulation of the thesis and this is as follows: analysis of optimal driving regimes by applying Pontryagin's Maximum principle (PMP), and proofs of switching times between optimal driving regimes for minimum-time, and minimum-time-energy problems. By performing additional calculations, we set non-linear programming problem. In section 5, we provide the numerical experiments of this non-linear programming problem in detail for minim-time, minimum-energy, and minimum-time-energy problems. The final section contains conclusions and open research problems.

2 A Survey of Basic Mathematical Models Used in Energy-Efficient Train Controls

In this survey, we divide the problem of driving train along the track into subproblems [14]:

- The problem of driving the train along level track (zero track gradient).
- The problem of driving the train along non-level track (non-zero track gradient).

The energy-efficient train control problem can be modeled [14] by:

- The electric energy consumption model.
- The fuel consumption model.

2.1 Review of the Electric Energy Consumption Model

2.1.1 Review a Formulation of the Electric Energy Consumption Model of Driving Train Along the Level Track

We consider the problem of driving an electric train along the level track from one station to the next station within a given allowable time T in such a way that an electric energy consumption is minimized [18]. Let 0 and s be the initial and terminal positions of the train. The speed of the train is 0 at the origin and destination station. The train speed $v(t)$ at time t is governed by a tractive function $F(t)$ and resistance force $R(v)$ according to the Newtonian laws of motion [4]

$$\dot{x}(t) = v(t), \tag{2.1.1.1}$$

$$m\dot{v}(t) = F(t) - R(v(t)), \tag{2.1.1.2}$$

where $x(t)$ is the position of the train at a time t, and m is the mass of the train. If we divide through by the mass of the train the above (2.1.1.2) can be written as follows:

$$\dot{v}(t) = u(t) - r[v(t)], \tag{2.1.1.3}$$

where $u(t)$ is the acceleration along the level track corresponding to the applied force $F(t)$, $-r[v(t)]$ is the resistive acceleration along the level track where $r[v(t)] = r_0 + r_1 v(t) + r_2 v^2(t)$ is given by the *Davis formula* [9].

The mathematical model of an optimal train control problem along level tracks when $T > 0$ is not a fixed, is represent a minimum time give by

$$T \to min \tag{2.1.1.4}$$

subject to

$$\dot{x}(t) = v(t), \tag{2.1.1.5}$$

$$\dot{v}(t) = u(t) - r(v(t)), \tag{2.1.1.6}$$

$$X(0) = 0, \qquad x(T) = s, \tag{2.1.1.7}$$

$$v(0) = 0, \qquad v(T) = 0. \tag{2.1.1.8}$$

The positive part of applied force can written as a piecewise function:

$$F^{+}(t) = \begin{cases} F(t), & F(t) > 0, \\ 0, & F(t) \leq 0, \end{cases}$$

and the negative part of applied force is also written as:

$$F^{-}(t) = \begin{cases} 0, & F(t) > 0, \\ -F(t), & F(t) < 0, \end{cases}$$

where $F(t) > 0$ is energy supplied to the train, and $F(t) < 0$ is energy dissipated by brakes. We can define the applied acceleration corresponding to the applied force $F(t)$ as:

$$u^{+}(t) = \begin{cases} u(t), & u(t) > 0, \\ 0, & u(t) \leq 0, \end{cases}$$

and the negative part of applied force is given as:

$$u^{-}(t) = \begin{cases} 0, & u(t) > 0, \\ -u(t), & u(t) < 0. \end{cases}$$

The electric energy consumption to be minimized is the total energy supplied to the train, and it is given by work done by the traction power $P(t) = F^+(t)v(t)$ over time, it is equal to $\int u^+(t)v(t)dt$.

Thus, we can formulate the basic optimal train control problem with a fixed time $T > 0$ along the level track with a minimum consumption of electric energy as [19]:

$$J(u, v) = \int_0^T u^+(t)v(t)dt \rightarrow min \tag{2.1.1.9}$$

subject to

$$\dot{x}(t) = v(t), \tag{2.1.1.10}$$

$$\dot{v}(t) = u(t) - r(v(t)), \tag{2.1.1.11}$$

$$x(0) = 0, \qquad x(T) = s, \tag{2.1.1.12}$$

$$v(0) = 0, \qquad v(T) = 0, \tag{2.1.1.13}$$

$$v(t) \geq 0, \quad u(t) \in [-u_{min}, u_{max}(v(t))], \tag{2.1.1.14}$$

where $x(t)$ is the distance traveled over time t, and s is the total distance traveled. The variables (x, v) are the state variables and u is control variable.

If we consider the same problem (2.1.1.9)-(2.1.1.14), but, when criterion of optimality involves the braking of the train, the objective function is given by

$$\int_0^T (u^+(t) - \eta u^-(t))v(t)dt \rightarrow min, \tag{2.1.1.15}$$

where $\eta\ u^-(t)v(t)$ is the energy regenerated by the braking of the train, and $\eta \in [0, 1]$ represents the portion of the electric energy that is being reloaded to the electric circuit (grid) while braking.

If we consider the same problem when criterion of optimality involves both previous view of point that means the time duration T and an electric energy consumption in such a case, we can write the criterion condition in a joined form

$$\int_0^T (p_1 u^+(t)v(t) + p_2)dt \rightarrow min, \tag{2.1.1.16}$$

where p_1, $p_2 > 0$ is a real numbers such that $p_1 + p_2 = 1$, are called weighted parameters.

2.1.2 Review a Formulation of the Electric Energy Consumption Model of Driving Train Along the Non-Level Track

We assume that both gradients and speed limits are functions of distance. Hence, x is independent variable. The state variables become $t(x)$ and speed $v(x)$. Let us assume that $u(x) \in U = [-u_{min}, u_{max}(v(x))]$ for $x \in [0, s]$. Hence, an electric energy consumption to be minimized is given by [19]:

$$J(x) = \int_0^X u^+(x)dx \to min \tag{2.1.2.1}$$

subject to

$$\dot{t}(x) = \frac{1}{v(x)}, \tag{2.1.2.2}$$

$$v(x)\dot{v}(x) = (u(x) - r(v(x)) - g(x)), \tag{2.1.2.3}$$

$$t(0) = 0, \qquad t(x) = T, \tag{2.1.2.4}$$

$$v(0) = 0, \qquad v(X) = 0, \tag{2.1.2.5}$$

$$v(x) \in [0, v_{max}(x)], \tag{2.1.2.6}$$

where T is total time, and (t, v) are state variables. We can assume that the non-level tracks have a piecewise gradient that means $g(x) > 0$ on up-hills slopes, and $g(x) < 0$ on downhill slopes. $g(x) = -gsin\theta(x)$, where g is acceleration due to gravity, and $\theta(x)$ is the angle of slope at a distance x along the track. Therefore, the total resistance can be give as

$$r(v, x) = r(v) + g(x) = r_2v^2 + r_1v + r_0 + g(x), \tag{2.1.2.7}$$

where r_2, r_1, and r_0 are nonnegative coefficients.

If we consider the same problem (2.1.2.1)-(2.1.2.6), but, when criterion of optimality involves the braking of the train, the objective function is given by

$$\int_0^T (u^+(x) - \eta u^-(x))dx \to min, \tag{2.1.2.8}$$

where $\eta u^-(x)$ is the energy regenerated by the braking of the train.

2.2 Review of the Fuel Consumption Model

2.2.1 Review of a Formulation of the Fuel Consumption Model on Level Track

The feasible strategy that minimize the fuel consumption described as the energy-efficient train control problem with discrete control setting can be formulated [6] as follows.

Let us assume that there are $m+1$ district control settings f_j, j= 0,1, ...m be the fuel supply rate corresponding to the control setting j with f_0= 0 the zero fuel case corresponding to coasting, and $f_j < f_{j+1}$, j=1...m a sequence of increasing fuel supply rates. Moreover, let $\{t_k\}_{k=1,2,\dots,n}$ be the switching times, t_{n+1} is stopping time with $f_{j_{k+1}}$ the rate of the fuel supply maintained in the interval (t_k, t_{k+1}) for a duration of $\tau_{k+1}= t_{k+1} - t_k$. Let $t_0 = 0$ and $t_{n+1} = T$. Furthermore, we assume that braking is only applied at the final stage with the maximum braking rate b. Then, the minimum fuel consumption problem is modeled as

$$J = \sum_{k=0}^{n} f_{j(k+1)} \tau_{k+1} \to min \tag{2.2.1.1}$$

subject to

$$\dot{v}(t) = \frac{H f_{j(k+1)}}{v(t)} - r(v(t)), \qquad t \in [t_k, t_{k+1}), \tag{2.2.1.2}$$

$$\dot{v}(t) = b - r(v(t)), \qquad t \in [t_n, t_{n+1}], \tag{2.2.1.3}$$

$$\dot{x}(t) = v(t), \qquad v(t) \geq 0, \tag{2.2.1.4}$$

$$x(0) = 0, \qquad x(T) = s, \tag{2.2.1.5}$$

$$v(0) = 0, \qquad v(T) = 0, \tag{2.2.1.6}$$

where H is a constant. We assume that $r(0) > 0$, $r(v)$ strictly nondecreasing and the function $vr(v)$ is strictly convex.

2.2.2 Formulation of the Fuel Consumption Model on Non-Level Track

It is often reasonable to consider x as independent variable since both gradients and speed limits are functions of x. Then, the equation of the motion of a point mass train is formulated formulated as follows.

$$\dot{t}(x) = \frac{1}{v(x)} \tag{2.2.2.1}$$

and

$$v(x)v\dot{(}x) = \begin{cases} \frac{Hf_j}{v(x)} - r(v(x)) + g(x), & j \geq 0, \\ -k(j) - r(v(x)) + g(x), & j < 0, \end{cases}$$

where $j = j(x)$ the control setting, f_j is the fuel supply rate for control setting j, H is a constant proportionality and $k(j)$ is the braking acceleration for control j. When $j = 0$ corresponds to coasting. The fuel consumption is given by

$$J(x) = \int_0^X \frac{f(x)}{v(x)} dx, \tag{2.2.2.2}$$

where $f(x) = f_{j(x)}$ is the fuel supply rate and s is the total distance traveled.

3 Basic Concepts of Optimal Control Theory and Pontryagin's Maximum Principle

We investigate the behavior of an object whose immediate state is given by n-tuples $(x_1, x_2, ..., x_n) \in X \subset R^n$, where X is called a *phase space*. The coordinate $x(t) = (x_1(t), x_2(t), ..., x_n(t))$ is the position of the velocity of an object which is time dependent. Furthermore, we assume the motion of the object can be controlled from the mathematical point of views, they are given $k \in N$, $u_1, u_2, ..., u_k$ with the values from the set $U \subset R^k$ that may be influence motion of the object (there are time dependent). The behavior of the object is assumed to be prescribed by the system of n ordinarily differential equations (ODEs)

$$\dot{x} = f(x, u), \tag{3.0.2.3}$$

where $x = (x_1, x_2, ..., x_n) \in X$, $u = (u_1, u_2, ..., u_k) \in U$, $f = (f_1, f_1, ..., f_n)$. This system is called the *controlled system*.

Definition: A vector variable $u(t) = (u_1(t), u_2(t), ..., u_k(t))$ defined on $[t_0, t_n]$ with values $U \in R^k$ is called *control*. The set U is called the *range of the control*.

Restriction on U: We assume it is a compact (finite and closed).

Restriction on f: f satisfies assumptions that ensured the existence and uniqueness of the solution to the system (3.0.2.3) started with from the initial condition

$$x(t_0) = a \in X. \tag{3.0.2.4}$$

For more detail, see [1], [3], [7], and [15].

For the restriction on u, we consider the followings.

Definition: We introduce the class of feasible control if their elements satisfy the following properties [8]:

Property 1: $u(t)$, $t \in I$ is a feasible control, then $u(t)$, $t \in J \subset I$ is also a feasible control (i.e., any part of feasible control again a feasible control).

Property 2: If $u_1(t)$, $t \in [t_0, t_1]$ and $u_2(t)$, $t \in [t_1, t_2]$ are feasible control then also

$$\hat{u}(t) = \begin{cases} u_1(t), & t \in [t_0, t_1) \\ u_2(t), & t \in [t_1, t_2] \end{cases} \tag{3.0.2.5}$$

is again a feasible control (i.e., attachment of the feasible control is again a feasible control).
Property 3: If u(t), $t \in [t_0, t_1]$ is a feasible control, then for any $a \in X$ there exist a unique solution to the initial value problem (3.0.2.3) and (3.0.2.4).

3.1 The Choice of a Class of Feasible Control

The natural choice is a class of continuous function. Property 2 need not be satisfied automatically (because the attachment of two continuous functions might not be continuous). On the other hand, property 1 and property 3 are satisfied. A suitable compromise satisfying all three properties is the *class of piecewise continuous functions* (i.e., the class of functions that are continuous on the domain $[t_0, t_1]$ up to possible exception of finitely many points where the functions have finite but different one sided limit; we assume that there are right continuous on $[t_0, t_1]$). Considering this class all three properties are satisfied (in particular the existences and uniqueness of initial value problem (3.0.2.3)and (3.0.2.4) can be verified "by continuous part").
Remark 1: In all this thesis, we assume that a feasible control is a piecewise continuous function (more precisely, a right continuous on its domain $[t_0, t_1]$).

3.2 Formulation of a Basic Optimal Control Problem

Definition: Let $a, b \in X$. We say that a feasible control $u(t)$, $t \in [t_0, t_1]$ convert the point "a" into the point "b" if the corresponding solution of the differential system (3.0.2.3) satisfying the initial condition (3.0.2.4), and terminal condition

$$x(t_1) = b. \tag{3.2.0.6}$$

Definition: A couple $(u(t), x(t))$, $t \in [\, t_0,\, t_1]$ is called *controlled process converting* "a" to "b".

Remark 2: Since the system (3.0.2.3) is autonomous (i,e., it does not explicitly depend on the time (time-invariant system)), we can move controls along the time axis. More precisely, $u(t)$, $t \in [t_0, t_1]$ converts "a" to "b"then, the control $u(t+\gamma)$, $t \in [t_0 - \gamma, t_1 - \gamma]$ where $\gamma \in R$ has the same property. Thus, without loss of generality, we can put $t_0 = 0$ and $t_1 = T$. Through this thesis, we use [0,T].

Definition: Let $f_0(x, u)$ be an objective function satisfying the same assumptions as those put on $f(x, u)$. Let $a, b \in X$ be arbitrary. Among all feasible controls u converting "a" to "b" (if such control exist), we are looking for control $\hat{u}(t)$, $t \in [0, \hat{T}]$ such that the objective function

$$J = \int_0^T f_0(x(t), u(t))dt \tag{3.2.0.7}$$

is minimized just when $u = \hat{u}(t)$ and $\hat{T}$. This control $\hat{u}(t)$ is called *optimal control* and the corresponding trajectory $\hat{x}(t)$ is called *optimal trajectory* on $t \in [0,]$. The couple ($\hat{u}(t)$, $\hat{x}(t)$) is called *optimal control process*, $t \in [0, \hat{T}]$.

In short we can write as

$$\begin{aligned} \dot{x} &= f(x, u), \quad u \in U, \\ x(0) &= a \in X, \\ x(T) &= b \in X, \\ J &= \textstyle\int_0^T f_0(x, u)dt \to min. \end{aligned} \tag{3.2.0.8}$$

3.3 Pontryangin's Maximum Principle

In this section, we formulate the basic assertion of optimal control theory. For easier formulation, we introduce some additional symbols and variables. We consider the basic control system (3.0.2.3) extended by the additional zero equation $\dot{x_0} = f(x, u)$, where x_0 is objective function of our problem and we get the extended controlled system

$$\dot{x_0^*} = f^*(x, u),$$

where $x^* = (x_0, x_1, ..., x_n)$, $f^* = (f_0, f_1, ..., f_n)$ and $x = (x_1, x_2, ..., x_n)$. Furthermore, we introduce the differential system for other additional variables ψ_0, ψ_1,..., ψ_n in form

$$\begin{aligned}
\dot{\psi}_0 &= -\sum_{k=0}^{n} \frac{\partial f_k(x,u)}{\partial x_0}\psi_k, \\
\dot{\psi}_1 &= -\sum_{k=0}^{n} \frac{\partial f_k(x,u)}{\partial x_1}\psi_k, \\
&\vdots \\
\dot{\psi}_n &= -\sum_{k=0}^{n} \frac{\partial f_k(x,u)}{\partial x_n}\psi_k.
\end{aligned} \tag{3.3.0.9}$$

This system is called adjoint system and functions ψ_0, ψ_1,..., ψ_n are called *generalized Lagrange Multipliers* (the same as classical multipliers in optimization problem for functions of several variables). We denote ψ^*= (ψ_0, ψ_1,..., ψ_n) and introduce the Hamiltonian function as

$$H^* = \psi^* \cdot f^* = \sum_{k=0}^{n} \psi_k f_k. \tag{3.3.0.10}$$

We can write the extended control system $\dot{x_0}^*$= $f^*(x,u)$ and initial system for Lagrange Multipliers jointly in the form of the Hamiltonian system

$$\begin{aligned}
\dot{x}_i &= \frac{\partial H^*}{\partial \psi_i} \\
\dot{\psi}_i &= \frac{\partial H^*}{\partial x_i}, i = 0, 1, 2, ..., n.
\end{aligned} \tag{3.3.0.11}$$

Maximum Principle

We consider the optimal control problem in its basic form (3.2.0.8). For the free-time problem the Maximum Principle states as follow. Let T be a free value. Further, let $\hat{u}(t)$ be the optimal solution and $\hat{x}(t)$ be the corresponding optimal trajectory on $t \in [0, \hat{T}]$ then there exist such that continuous and non-zero solution of the adjoint system

$$\dot{\psi}_i = \frac{\partial H^*}{\partial x_i}(\psi^*, \hat{x}, \hat{u}), i = 0, 1, 2, ..., n, \tag{3.3.0.12}$$

on the interval $[0, \hat{T}]$. Then, H^* satisfies the maximum condition

$$\max_{u \in U} H^*(\psi^*(t), \hat{x}(t), u(t)) = H^*(\psi^*(t), \hat{x}(t), \hat{u}(t)). \tag{3.3.0.13}$$

Moreover, H^* (ψ^*,$\hat{x}$,$\hat{u}$) $\equiv 0$ on [0,$\hat{T}$] and $\psi_0(t) \equiv \psi_0 \leq 0$.

Remark 3:

1. The Maximum Principle is a necessarily condition for optimality of $\hat{u}$. It continuous and generalizes a view of similar optimization problems (for function of single variable, for function of several variables, and so on).

2. The main part of the assertion is the maximum condition (3.3.0.13). Also, the condition is only necessary, its strong enough to generate several optimal control candidates. In many cases, it generates even a unique candidate for optimal control.

3. A weaker form of the Maximum Principle $\frac{\partial H^*}{\partial u}|_{u=\hat{u}} = 0$. It looks perhaps easier than the maximum condition, but its practical potential is very limited.

4. The proof of the Maximum Principle is very long and omitted.

Now we discuss some particular cases:
The simplest one is a *time optimization problem*. Its general form is

$$\begin{aligned} \dot{x} &= f(x, u), u \in U \\ x(0) &= a \in X \\ x(T) &= b \in X \\ T &\to min. \end{aligned} \tag{3.3.0.14}$$

The time optimization problem is obtained from (3.0.2.3) by the choice $f_0 = 1$. In this case, the Hamiltonian function has the form

$$H = \psi \cdot f = \psi_0 + \psi_1 f_1 + \ldots + \psi_n f_n. \tag{3.3.0.15}$$

Since ψ_0 is constant (independent of u) the Hamiltonian function is maximized if and only if the reduced Hamiltonian form,

$$H = \psi \cdot f = \psi_1 f_1 + \ldots + \psi_n f_n \tag{3.3.0.16}$$

is maximized over u.

3.3.1 Maximal Principle for Time Optimization Problem

Let $\hat{u}(t)$ be the optimal solution of the problem (3.3.0.14) and $\hat{x}(t)$ be the corresponding optimal trajectory on $t \in [0,\ \hat{T}]$. Then, there exist a continuous and non-zero solution $\psi = (\psi_1, \psi_2 \ldots, \psi_n)$ of the adjoint system

$$\dot{\psi_i} = -\frac{\partial H}{\partial x_i}(\psi, \hat{x}, \hat{u}), i = 1, 2, ..., n \tag{3.3.1.1}$$

on $[0,\ \hat{T}]$ such that $H = \psi \cdot f = \psi_0 + \psi_1 f_1 + \ldots + \psi_n f_n$ satisfies the maximum condition

$$\max_{u \in U} H(\psi, \hat{x}, u) = H(\psi, \hat{x}, \hat{u}). \tag{3.3.1.2}$$

Moreover, the reduced Hamiltonian $H(\psi, \hat{x}, \hat{u}) \equiv -\psi_0 \geq 0$ on $[0, \hat{T}]$, $\psi_0 < 0$. Now, we consider the optimal control problem(3.0.2.3) when T is fixed. We extend the dynamical system $\dot{x}$=$f(x, u)$ by additional n+1st equation $\dot{x_{n+1}} = 1$ supported by the initial condition $x_{n+1}(0)$=0. Then, x_{n+1}= t (i.e., x_{n+1} is a time). Let $\widetilde{T}$ be known prescribed value of the time. We consider the optimal control with a free time T converting (a, 0) (the initial state) to (b, $\widetilde{T}$). This implies the free value of T must be equal to $\widetilde{T}$ (prescribed values of $\widetilde{T}$).

3.3.2 Maximum Principle for a Fixed Time Problem

We consider the problem(3.2.0.8), where the value of T is fixed. Let $\hat{u}(t)$ is the optimal solution of this problem and $\hat{x}(t)$ be the corresponding optimal trajectory on $t \in [0, \hat{T}]$. Then, it holds the conclusion of the Maximum Principle for free time except to

$$H^*(\psi^*, \hat{x}, u) = C$$

on $[0, \hat{T}]$, which is replaced by $H^*(\psi^*, \hat{x}, u) = 0$ on $[0, \hat{T}]$.

For further information on optimal control theory, we refer to [5], [10], [11], [13], [16], and [17].

4 The Main Formulation of the Problem: Results and Discussion

We consider the problem of driving an electric train from station A to the next station B. Our aim is to drive the train along the level track into the station B in shortest possible time T with a minimum consumption of electric energy. We take into the considerations also the air resistance and assume it depends on the train velocity.

A Mathematical Formulation

Let $y = y(t)$ be a position of train and we place the origin of y-axis into the station A and s be the distance between two stations.

Figure 1: Illustration of the main problem

Furthermore, let m be the mass of the train, and $r = r(v)$ be the resistance function depending on the train velocity v.

Assumption on r:

$$r \in C^2([0,\infty)), \quad r(0) = 0, \quad r(v) > 0 \ \text{ for } \quad v > 0,$$

$$r'(0) \geq 0, \quad r'(v) > 0 \ \text{ for } \ v > 0,$$

$$r''(v) \geq 0 \ \text{ for } \ v \geq 0.$$

This general assumptions involve all standard resistance functions such that $r(v) = kv$, $v > 0$ (linear function) and $r(v) = kv^2$, $v > 0$ (quadratic function).

Based on our aim, we are combining minimum-time, minimum-energy, and minimum-time-energy problems. Therefore, we can use the real weight parameters p_1, $p_2 \geq 0$ such that

$p_1 + p_2 = 1$. Hence, we can write our objective function in a joined form

$$J = \int_0^T (p_1 u_\eta \dot{y} + p_2) dt \to min \tag{4.0.2.1}$$

subject to

$$m\ddot{y} = u - r(\dot{y}), \quad U = [-\alpha, \beta], \quad \alpha, \beta > 0, \tag{4.0.2.2}$$

$$y(0) = \dot{y}(0) = 0, \tag{4.0.2.3}$$

$$y(T) = s, \quad \dot{y}(T) = 0, \tag{4.0.2.4}$$

$$u \in U. \tag{4.0.2.5}$$

Here, u_η is defined as

$$u_\eta = u^+(t) - \eta u^-(t) = \begin{cases} u(t), & u(t) \geq 0, \\ \eta u(t), & u(t) < 0, \end{cases}$$

where

$$u^+(t) = \begin{cases} u(t), & u(t) > 0, \\ 0, & u(t) \leq 0, \end{cases}$$

$$u^-(t) = \begin{cases} 0, & u(t) > 0, \\ -u(t), & u(t) < 0. \end{cases}$$

$\eta \in [0,1]$ represents the portion of the electric energy that is being reloaded to the electric circuit (grid) while braking. Without loss of generality, we put $m = 1$.

Finally, we rewrite this model into phase variables.

Since the Pontryagin's Maximum Principle is applicable only for first order differential system, we must convert this second order differential system to first order differential system. This can be done as follows.

Let $x_1 = y$, $x_2 = \dot{y}$, then

$$\dot{x_1} = x_2, \tag{4.0.2.6}$$

$$\dot{x_2} = u - r(x_2), \quad -\alpha \leq u \leq \beta, \tag{4.0.2.7}$$

$$x_1(0) = x_2(0) = 0, \tag{4.0.2.8}$$

$$x_1(T) = s, \quad x_2(T) = 0, \tag{4.0.2.9}$$

$$J = \int_0^T (p_1 u_\eta x_2 + p_2) dt \to min, \tag{4.0.2.10}$$

where

$$r \in C^2([0, \infty)), \quad r(0) = 0, \quad r(x_2) > 0 \text{ for } x_2 > 0,$$

$$r'(0) \geq 0, \quad r'(x_2) > 0 \text{ for } x_2 > 0,$$

$$r''(x_2) \geq 0 \quad \text{for} \quad x_2 \geq 0.$$

It is an optimal control problem with fixed time (when p_1=1), or with a free time (when $0 \leq p_1 < 1$).

Remark 4: The problem (4.0.2.6)-(4.0.2.10) depending on the weight parameters can be categorized as:

- The minimum-time problem when $p_1 = 0$, $p_2 = 1$.
- The minimum-energy problem, when $p_1 = 1$, $p_2 = 0$.
- The minimum-time-energy problem when $0 < p_1 < 1$ and $0 < p_2 < 1$.
- The detail explanation will be given later.

4.1 Minimum-Time Problem: Results and Discussion

4.1.1 Application of Pontryagin's Maximum Principle for Minimum-Time Problem

According to the Pontryagin's Maximum Principle (PMP) the optimal control variables $\hat{u}$ should be selected from the admissible control variables that maximize Hamiltonian function. The minimum-time problem is the case when $\psi_0 = C_0 = 0$, i.e., $p_1 = 0$, $p_2 = 1$ (see section 3.3.1 and Remark 4) then, the extended Hamiltonian $H^* = \psi_0(p_1 u_\eta x_2 + p_2) + \psi_1 x_2 + \psi_2(u - r(x_2))$ is reduce as

$$H = \psi_1 x_2 + \psi_2(u - r(x_2)). \tag{4.1.1.1}$$

Let $\hat{u}(t)$ be optimal control and $\hat{x}(t)$ be the corresponding optimal trajectory on $t \in [0, \hat{T}]$. Then, the maximum condition yields,

$$\max_{-\alpha \le u \le \beta} \psi_1 \hat{x_2} + \psi_2(u - r(\hat{x_2})) = \psi_1 \hat{x_2} + \psi_2(\hat{u} - r(\hat{x_2})) \text{ on } [0, \hat{T}].$$

We eliminate the terms which are not depending on the control. Then, we have

$$\max_{-\alpha \le u \le \beta} \psi_1 \hat{x_2} + \psi_2(u - r(\hat{x_2})) = \psi_2 \hat{u} \text{ on } [0, \hat{T}]. \tag{4.1.1.2}$$

From (4.1.1.2), we can deduce 3 possible variances for $\hat{u}(t)$ as follows:

$$\hat{u}(t) = \begin{cases} \beta, & \psi_2(t) > 0, \\ ?, & \psi_2(t) = 0, \\ -\alpha, & \psi_2(t) < 0. \end{cases} \tag{4.1.1.3}$$

The adjoint system is given

$$\dot{\psi_1} = -\frac{\partial H^*}{\partial x_1} = 0 \Rightarrow \psi_1 = C_1, \tag{4.1.1.4}$$

$$\dot{\psi_2} = -\frac{\partial H^*}{\partial x_2} = -\psi_1 + \psi_2 r'(x_2). \tag{4.1.1.5}$$

From (4.1.1.3), the value of $\hat{u}(t)$ is not specified by the Maximum Principle for $\psi_2(t) = 0$. This case is usually referred to as the the *singular case*. We show that the undetermined (singular) case cannot be occurred, that means, the appropriate equality can occur only in isolated points.

Assume on the contrary that $\psi_2(t) \equiv 0$ on some nontrivial interval $J \subset [0, \hat{T}]$. This implies $\dot{\psi_2}(t) \equiv 0$ on J and (4.1.1.5) implies that

$$-\psi_1 + \psi_2 r'(x_2) \equiv 0 \text{ on J}.$$

Since $\psi_2(t) \equiv 0$ by assumption, we have $C_1 = 0$. Consequently, (4.1.1.5) becomes $\dot{\psi_2}(t) = \psi_2 r'(x_2)$ on $[0, \hat{T}]$ because $C_1 = 0$ on the whole domain. If $\psi_2(t^*) > 0$ then, this implies $\dot{\psi_2}(t^*) > 0$ for any t^*. This means that, if there exists $t^* \in [0, T] - J$ such that $\psi_2(t^*) > 0$ then, $\psi_2(t) > 0$ for all $t \ge t^*$.

On the other side, it must hold that $\psi_2(0) > 0$ because in the opposite case, we are not able to leave the station A. By the previous conclusion, $\psi_2(0) > 0$ implies that $\psi_2(t) > 0$ for all $t \in [0, \hat{T}]$. But, it contradicts our assumption $\psi_2(t) \equiv 0$ on nontrivial interval J. In other words, $\psi_2(t) = 0$ can occur only in isolated time t, that is not on a nontrivial interval. We actually eliminate the case $\hat{u}(t)$ is undermined if $\psi_2(t) \equiv 0$ (since such a control cannot be optimal one).

Thus, we have found two optimal driving regimes

$$\hat{u}(t) = \begin{cases} \beta, & \psi_2(t) > 0, \\ -\alpha, & \psi_2(t) < 0. \end{cases} \tag{4.1.1.6}$$

The next step is to prove that optimal driving regimes appear in order given as (4.1.1.6) and they cannot be repeated.

The proof of this claim depends on the sign analysis of $\psi_2(t)$. Clearly $\psi_2(0) > 0$. Assume that $\psi_2(t) > 0$ for all $0 \leq t \leq \hat{T}$. Then, $\hat{u}(t) = \beta$ for all $0 \leq t \leq \hat{T}$ and this contradicts the condition $x_2(\hat{T}) = 0$. That means, there exists $0 < t_s < \hat{T}$ such that $\psi_2(t) > 0$ for all $0 \leq t < t_s$ and $\psi_2(t_s) = 0$. By the Mean Value Theorem (MVT), $\dot{\psi}_2(t^*) < 0$ for some $0 < t^* < t_s$. This implies that $\psi_1 \equiv \psi_1 > 0$ by (4.1.1.5). Therefore, $\dot{\psi}_2(t_s) = -\psi_1 < 0$ and $\psi_2(t) < 0$ for all $t = t_s + \epsilon$, where $\epsilon > 0$ is sufficiently small. Hence, $\psi_2 < 0$ for all $t_s < t < \hat{T}$. In opposite case, there exists $t_s < t^{**} \leq \hat{T}$ such that $\psi_2(t^{**}) < 0$ and $\dot{\psi}_2(t^{**}) = 0$. This is a contradiction. From this, we get $\psi_2(t) < 0$ for all $t_s < t < \hat{T}$.

Thus, we can summarize the result as: there exists t_s such that $0 < t_s < \hat{T}$,

$$\hat{u}(t) = \begin{cases} \beta, & t \in [0, t_s) \\ -\alpha, & t \in \left[t_s, \hat{T}\right]. \end{cases} \tag{4.1.1.7}$$

We call $\hat{T}$ as T_{min} (its value will be specified later).

Remark 5: To determine the values of T_{min} and t_s, it is necessarily to take the maximum acceleration ($\hat{u}(t) = \beta$) and the maximum braking ($\hat{u}(t) = -\alpha$). This will be done later, after specifying the resistance function r.

4.2 Minimum-Time-Energy Problem: Results and Discussion

In this section, we analyse the minimum-time-energy problem. We emphasise that the next argumentation is valid for the corresponding minimum-energy problem as well.

4.2.1 Application of Pontryagin's Maximum Principle for Minimum-Time-Energy Problem

According to the PMP the optimal control variables $\hat{u}$ should be selected from the admissible control variables that maximize the extended Hamiltonian function

$$H^* = \psi_0(p_1 u_\eta x_2 + p_2) + \psi_1 x_2 + \psi_2(u - r(x_2)). \tag{4.2.1.1}$$

Let $\hat{u}(t)$ be optimal control and $\hat{x}(t)$ be the corresponding optimal trajectory on $t \in [0, \hat{T}]$. Then, the maximum condition yields

$$\max_{-\alpha \le u \le \beta} \psi_0(p_1 u_\eta \hat{x_2} + p_2) + \psi_1 \hat{x_2} + \psi_2(u - r(\hat{x_2})) = \psi_0(p_1 \hat{u}_\eta \hat{x_2} + p_2) + \psi_1 \hat{x_2} + \psi_2(\hat{u} - r(\hat{x_2})) \text{ on } \left[0, \hat{T}\right]. \tag{4.2.1.2}$$

We eliminate the terms which are not depending on the control. This gives:

$$\max_{-\alpha \le u \le \beta} \psi_0(p_1 u_\eta \hat{x_2} + p_2) + \psi_1 \hat{x_2} + \psi_2(u - r(\hat{x_2})) = \psi_0 p_1 \hat{u}_\eta \hat{x_2} + \psi_2 \hat{u} \text{ on } \left[0, \hat{T}\right]. \tag{4.2.1.3}$$

By the definition of u_η, we split the maximum condition into two relations.
For $u \ge 0$:

$$\max_{0 \le u \le \beta} \psi_0(p_1 u \hat{x_2} + p_2) + \psi_1 \hat{x_2} + \psi_2(u - r(\hat{x_2})) = \hat{u}(\psi_0 p_1 \hat{x_2} + \psi_2) \text{ on } \left[0, \hat{T}\right]. \tag{4.2.1.4}$$

For $u \le 0$:

$$\max_{-\alpha \le u \le 0} \psi_0(p_1 \eta u \hat{x_2} + p_2) + \psi_1 \hat{x_2} + \psi_2(u - r(\hat{x_2})) = \hat{u}(\psi_0 p_1 \eta \hat{x_2} + \psi_2) \text{ on } \left[0, \hat{T}\right]. \tag{4.2.1.5}$$

From these two relations, we can deduce 5 possible variances for $\hat{u}(t)$ as follows:

$$\hat{u}(t) = \begin{cases} \beta, & \psi_0 p_1 \hat{x_2}(t) + \psi_2(t) > 0, \\ ?, & \psi_0 p_1 \hat{x_2}(t) + \psi_2(t) = 0, \quad \text{undetermined on } [0, \beta], \\ 0, & \psi_0 p_1 \hat{x_2}(t) + \psi_2(t) < 0, \quad \psi_0 p_1 \eta \hat{x_2}(t) + \psi_2(t) > 0, \\ ?, & \psi_0 p_1 \eta \hat{x_2}(t) + \psi_2(t) = 0, \quad \text{undetermined on } [-\alpha, 0], \\ -\alpha, & \psi_0 p_1 \eta \hat{x_2}(t) + \psi_2(t) < 0. \end{cases} \tag{4.2.1.6}$$

We can see that from (4.2.1.6) the case $\hat{u}(t) = 0$ cannot occur for $\eta = 1$. From (4.2.1.6), there are two singular cases. In the sequel, we investigate these singular cases.
The adjoint system from (4.2.1.1) is given

$$\begin{aligned} \dot{\psi_0} &= 0 \Rightarrow \psi_0 = C_0, \\ \dot{\psi_1} &= -\tfrac{\partial H^*}{\partial x_1} = 0 \Rightarrow \psi_1 = C_1, \\ \dot{\psi_2} &= -\frac{\partial H^*}{\partial x_2} = -\psi_0 p_1 u_\eta - \psi_1 + \psi_2 r'(x_2). \end{aligned} \tag{4.2.1.7}$$

We show that two undetermined cases cannot be occurred, i.e., the appropriate equality can occur only in isolated points.
Assume that $\psi_0 p_1 \eta \hat{x_2} + \psi_2 \equiv 0$ on a nontrivial interval $J \subset [0, \hat{T}]$. By taking the derivative with respect to time, we get

$$\psi_0 p_1 \eta \dot{\hat{x_2}} + \dot{\psi_2} \equiv 0 \quad \text{on J.} \tag{4.2.1.8}$$

Substituting $\dot{x_2} = u - r(x_2)$ and $\dot{\psi_2} = -\frac{\partial H^*}{\partial x_2} = -\psi_0 p_1 u_\eta - \psi_1 + \psi_2 r^{'}(x_2)$ into equation(4.2.1.8), we get

$$C_0 p_1 \eta (u - r(x_2)) - C_0 p_1 u_\eta - C_1 + \psi_2 r^{'}(x_2) \equiv 0 \quad \text{on J.} \tag{4.2.1.9}$$

From the Maximum Principle, $\psi_0 = C_0 \leq 0$. Since the case $\psi_0 = C_0 = 0$ corresponds the minimum-time problem, i.e., $p_1 = 0$, $p_2 = 1$ (see section 4, remark 4), we consider the case $\psi_0 = C_0 < 0$ (without loss of generality $C_0 = -1$, see section 3.3).
Singular case 1: Let $\hat{u} \in [-\alpha, 0]$ and $\hat{u}_\eta = \hat{u}\eta$ where $\hat{u} < 0$.
We assume that $0 < \eta \leq 1$ and $\psi_2 - p_1 \eta \hat{x_2} \equiv 0$ for $t \in J$, where $J \subset \left[0, \hat{T}\right]$ is a nontrivial interval.

From (4.2.1.9) we get

$$p_1\eta r(x_2)) - C_1 + \psi_2 r^{'}(x_2) \equiv 0 \text{ on J.} \tag{4.2.1.10}$$

We repeatedly derivate with respect to time. Then,

$$\frac{d}{dt}(r'(\hat{x}_2)) = r''(\hat{x}_2)\dot{\hat{x}}_2, \tag{4.2.1.11}$$

using (4.2.1.11), from (4.2.1.10) we get

$$\dot{\psi}_2 r'(\hat{x}_2) + \psi_2 r''(\hat{x}_2)\dot{\hat{x}}_2 + \eta p_1 r'(\hat{x}_2)\dot{\hat{x}}_2 \equiv 0. \tag{4.2.1.12}$$

From the assumption $\psi_2 - \eta p_1 \hat{x_2} \equiv 0$ on J, we have $\psi_2 \equiv \eta p_1 \hat{x_2}$ and $\dot{\psi_2} \equiv \eta p_1 \dot{\hat{x_2}}$ on J. Substituting these into (4.2.1.12) give

$$p_1\eta\dot{\hat{x_2}} r'(\hat{x}_2) + \eta p_1 \hat{x_2} r''(\hat{x}_2)\dot{\hat{x_2}} + \eta p_1 r'(\hat{x}_2)\dot{\hat{x_2}} \equiv 0. \tag{4.2.1.13}$$

By rewriting (4.2.1.13), we obtain

$$\dot{\hat{x_2}}(\eta p_1(2r'(\hat{x}_2) + \hat{x}_2 r''(\hat{x}_2))) \equiv 0 \text{ on J.}$$

Using $\dot{\hat{x_2}} = \hat{u} - r(\hat{x}_2)$, we get

$$(\hat{u} - r(\hat{x}_2))(\eta p_1(2r'(\hat{x}_2) + \hat{x}_2 r''(\hat{x}_2)) \equiv 0 \text{ on J.} \tag{4.2.1.14}$$

Due to the assumptions of r', r'' and also $p_1, \eta > 0$, we have $(2r'(\hat{x}_2) + \hat{x}_2 r''(\hat{x}_2)) > 0$. Hence, from (4.2.1.14) we get

$$\hat{u} - r(\hat{x_2}) \equiv 0 \text{ on J.}$$

$$\hat{u} \equiv r(\hat{x_2}) \text{ on J.} \tag{4.2.1.15}$$

We can see that (4.2.1.15) cannot occur on a nontrivial interval J because the optimal control, in this case, cannot be identically equal to the resistance function $r(\hat{x_2})$ (i.e., the optimal control is negative in this case whereas the resistance function is nonnegative). Therefore, this singular cannot occur for $0 < \eta \leq 1$.

Now, let us assume that $\eta = 0$. That means, $\psi_2 \equiv 0$ on a nontrivial interval $J \subset [0, \hat{T}]$. This implies $\dot{\psi}_2(t) \equiv 0$ on J and (4.2.1.7) implies that

$$-\psi_1 + \psi_2 r'(x_2) \equiv 0 \text{ on J.}$$

Since $\psi_2(t) \equiv 0$ we have $C_1 = 0$. Consequently, (4.2.1.7) becomes

$$\dot{\psi}_2(t) = \psi_2 r'(x_2) \quad \text{on}[0, \hat{T}] \tag{4.2.1.16}$$

because $C_1 = 0$ on the whole domain. If $\psi_2(t^*) > 0$, this implies $\dot{\psi}_2(t^*) > 0$ for any t^*. This means that, if there exists $t^* \in [0, T] - J$ such that $\psi_2(t^*) > 0$ then, $\psi_2(t) > 0$ for all $t \geq t^*$. On the other side, it must hold that $\psi_2(0) > 0$ because in the opposite case, we are not able to leave the station A. By the previous conclusion, $\psi_2(0) > 0$ implies that $\psi_2(t) > 0$ for all $t \in [0, \hat{T}]$. But, it contradicts our assumption $\psi_2(t) \equiv 0$ on nontrivial interval J. In other words, $\psi_2(t) = 0$ can occur only in isolated time t, that is not on a nontrivial interval. We actually eliminate the case $\hat{u}(t)$ is undermined if $\psi_2(t) \equiv 0$ since such a control cannot be optimal one.

In conclusion, *singular case 1* cannot occur provided that $0 \leq \eta \leq 1$.

Singular case 2: $\hat{u} \in [0, \beta]$. We assume that $0 \leq \eta \leq 1$ and $\psi_2 - p_1\hat{x_2} \equiv 0$ for $t \in J$, where $J \subset \left[0, \hat{T}\right]$ is a nontrivial interval. When $\eta = 1$, the two singular cases are merged, but here $\hat{u}$ is undetermined on $[0, \beta]$, $\hat{u}_\eta = \hat{u}$ where $\hat{u} > 0$. We can rewrite (4.2.1.9) as

$$-p_1(\hat{u} - r(\hat{x}_2)) + p_1\hat{u} - C_1 + \psi_2 r'(\hat{x}_2) \equiv 0 \quad \text{on J.} \tag{4.2.1.17}$$

We can simplify (4.2.1.17) as

$$p_1 r(\hat{x}_2) - C_1 + \psi_2 r'(\hat{x}_2) \equiv 0 \quad \text{on J.} \tag{4.2.1.18}$$

We repeatedly derivate with respect to time as above. Using (4.2.1.11), from (4.2.1.18) we get

$$\dot{\psi}_2 r'(\hat{x}_2) + \psi_2 r''(\hat{x}_2)\dot{\hat{x}}_2 + p_1 r'(\hat{x}_2)\dot{\hat{x}}_2 \equiv 0. \tag{4.2.1.19}$$

From the assumption $\psi_2 - p_1\hat{x_2} \equiv 0$ on J, we have $\psi_2 \equiv p_1\hat{x_2}$ and $\dot{\psi}_2 \equiv p_1\dot{\hat{x_2}}$ on J. Substituting these into (4.2.1.19) give

$$p_1\dot{\hat{x_2}} r'(\hat{x}_2) + p_1\hat{x_2} r''(\hat{x}_2)\dot{\hat{x_2}} + p_1 r'(\hat{x}_2)\dot{\hat{x_2}} \equiv 0.$$

From this, we get

$$\dot{\hat{x_2}}(p_1(2r'(\hat{x}_2) + \hat{x}_2 r''(\hat{x}_2))) \equiv 0 \qquad \text{on J.} \tag{4.2.1.20}$$

Since $\dot{\hat{x_2}} = \hat{u} - r(\hat{x}_2)$ for this case, from (4.2.20) we get

$$(\hat{u} - r(\hat{x}_2))(p_1(2r'(\hat{x}_2) + \hat{x}_2 r''(\hat{x}_2)) \equiv 0 \quad \text{on J.} \tag{4.2.1.21}$$

Due to the assumptions of r' and r'', $(2r'(\hat{x}_2) + \hat{x}_2 r''(\hat{x}_2)) > 0$ and also $p_1 > 0$. Hence, from (4.2.21) we get

$$\hat{u} \equiv r(\hat{x_2}) \ \text{ on } J.$$

To summarize, if the singular case $\psi_2 - p_1\hat{x_2} \equiv 0$ does occur on a nontrivial interval J then, $\hat{u} \equiv r(\hat{x_2})$. Hence, from all previous considerations, we have the following conclusion:

1. For the case $0 \leq \eta < 1$, we have found the following four optimal driving regimes:

$$\hat{u}(t) = \begin{cases} \beta, & \psi_2(t) - p_1\hat{x_2}(t) > 0, \\ r(\hat{x_2}), & \psi_2(t) - p_1\hat{x_2}(t) \equiv 0, \\ 0, & \psi_2(t) - p_1\hat{x_2}(t) < 0, \psi_2(t) - p_1\eta\hat{x_2}(t) > 0, \\ -\alpha, & \psi_2(t) - p_1\eta\hat{x_2}(t) < 0. \end{cases} \tag{4.2.1.22}$$

2. For the case $\eta = 1$, we have found the following three optimal driving regimes:

$$\hat{u}(t) = \begin{cases} \beta, & \psi_2(t) - p_1\hat{x_2}(t) > 0, \\ r(\hat{x_2}), & \psi_2(t) - p_1\hat{x_2}(t) \equiv 0, \\ -\alpha, & \psi_2(t) - p_1\hat{x_2}(t) < 0. \end{cases} \tag{4.2.1.23}$$

The next step is to show that the optimal driving regimes appear in order given as (4.2.1.22), (4.2.1.23), and they cannot be repeated (the background for some next procedures was taken from [21]).

1. **The case** $0 \leq \eta < 1$: From (4.2.1.22), the proof of this claim depends on the sign analysis of functions $\psi_2(t) - p_1\hat{x_2}(t)$ and $\psi_2(t) - p_1\eta\hat{x_2}(t)$ on $[0, \hat{T}]$. The values of $\hat{u}(t)$ depends on the relationship between $\psi_2(t)$ and $\hat{x_2}(t)$. These functions are continuous solution of

$$\begin{aligned} \dot{\psi}_2 &= p_1 u_\eta - \psi_1 + \psi_2 r'(x_2), \\ \dot{\hat{x_2}} &= \hat{u} - r(\hat{x_2}) \ \text{ on } \left[0, \hat{T}\right]. \end{aligned}$$

First, we show that if $\psi_2(t^*) = p_1\hat{x_2}(t^*)$ for some $0 \leq t^* < \hat{T}$ then, $\psi_2(t) \leq p_1\hat{x_2}(t)$ for all $t^* \leq t \leq \hat{T}$. On the opposite case, we assume that there exist $t'' > t' > t^*$ such that $\psi_2(t) > p_1\hat{x_2}(t)$ for all $t' < t < t''$. Thus,

$$\begin{aligned} \dot{\psi_2} &= p_1\beta - \psi_1 + \psi_2 r'(x_2), \\ \dot{\hat{x_2}} &= \beta - r(\hat{x_2}) \quad \text{on } (t', t''). \end{aligned} \tag{4.2.1.24}$$

We show that $\dot{\hat{x_2}}(t) > 0$ for all $t' < t < t''$. In other words, we verify that $\beta - r(\hat{x_2}) > 0$, that is, $\hat{x_2}(t) < r^{-1}(\beta)$ for these t. Clearly $\hat{x_2}(t') < r^{-1}(\beta)$. Now, assume that $\hat{x_2}(\tau) = r^{-1}(\beta)$ for some $t' < \tau < t''$. Then, the initial value problem

$$\begin{aligned} \dot{\hat{x_2}} &= \beta - r(\hat{x_2}), \\ \hat{x_2}(\tau) &= r^{-1}(\beta), \end{aligned}$$

admits two solutions on (t', t''), besides $x_2(t) = \hat{x_2}(t)$, $t' < t < t''$, it is a constant solution $x_2 = r^{-1}(\beta)$. This is a contradiction because the function r, r' are continuous on $[0, \infty)$. Therefore, $\dot{\hat{x_2}}(t) > 0$ for $t' < t < t''$. By (4.2.1.24), $\dot{\psi_2}$ is the increasing function of t and $\dot{\hat{x_2}}$ is decreasing function of t on (t', t''). This implies that, $\psi_2(t) > p_1\hat{x_2}(t)$ for all $t' < t \leq \hat{T}$ which contradicts the condition $x_2(\hat{T}) = 0$. Using this fact, we prove the order of switching times as given by (4.2.1.22) and they can not repeated.

The condition $x_2(0) = 0$ implies $\psi_2(0) > 0$ otherwise, we can not leave station A. In the opposite case, $\psi_2(0) \leq 0$ for all $0 < t \leq \hat{T}$ and $x_1(\hat{T}) = 0$. Then, there exists $0 < t_1 < \hat{T}$ such that $\psi_2(t) > p_1\hat{x_2}(t)$ for all $0 < t < t_1$ and $\psi_2(t_1) = p_1\hat{x_2}(t_1)$. Thus, there exist $t_1 \leq t < t_2 < t_3' < \hat{T}$ such that $\psi_2(t) = p_1\hat{x_2}(t)$ for $t_1 \leq t \leq t_2$ and $\eta p_1\hat{x_2}(t) < \psi_2(t) < p_1\hat{x_2}(t)$ for all $t_2 < t < t_3'$. By (4.2.1.22), $\hat{u(t)} = 0$ for $t_2 < t < t_3'$ and equations ψ_2, $\hat{x_2}$ on (t_2, t_3') become

$$\begin{aligned} \dot{\psi_2} &= -\psi_1 + \psi_2 r'(\hat{x_2}) \\ \dot{\hat{x_2}} &= -r(\hat{x_2}). \end{aligned}$$

The function $\dot{\psi_2}$ is decreasing and $\dot{\hat{x_2}}$ is increasing, hence there exists $t_3 > t_2$ such that $\eta p_1\hat{x_2}(t) < \psi_2(t) < p_1\hat{x_2(t)}$ for $t_2 < t < t_3$ and $\eta p_1\hat{x_2}(t_3) = \psi_2(t_3)$. Next, we show that $\psi_2(t) < \eta p_1 \hat{x_2}(t)$ for $t_3 < t \le \hat{T}$. Clearly $\psi_2(t+\epsilon) < \eta p_1 \hat{x_2}(t_3+\epsilon)$ for all ϵ sufficiently small. Then, we have

$$\begin{aligned} \dot{\psi_2} &= -p_1\eta\alpha - \psi_1 + \psi_2 r'(\hat{x_2}) \\ \dot{\hat{x_2}} &= -\alpha - r(\hat{x_2}) \quad \text{on} \quad \left(t_3, \hat{T}\right). \end{aligned}$$

Repeating the same arguments as above, we obtain $\psi_2(t) < p_1\eta\hat{x_2}(t)$ for all $t_3 < t \le \hat{T}$. Using the continuity of $\hat{u}$ from the right, $\hat{u}$ we can define $\hat{u}(t_1) = r(\hat{x_2}(t_1))$, $\hat{u}(t_2) = 0$, $\hat{u}(t_3) = -\alpha$ and this accomplish the proof of the case $0 \le \eta < 1$.

2. **The case** $\eta = 1$: From (4.2.1.23), the proof of this claim depends on the sign analysis of $\psi_2(t)$ and $p_1\hat{x_2}(t)$. The values of $\hat{u}(t)$ depends on the relationship between $\psi_2(t)$ and $\hat{x_2}(t)$. We recall that these functions are continuous solution of

$$\begin{aligned} \dot{\psi_2} &= p_1 u - \psi_1 + \psi_2 r'(x_2) \\ \dot{\hat{x_2}} &= \hat{u} - r(\hat{x_2}) \quad \text{on} \quad \left[0, \hat{T}\right]. \end{aligned}$$

Similarly as in the proof the case (1), we wish to show that there exist $0 < t_1 \le t_2 < \hat{T}$ such that

$$\begin{cases} \psi_2(t) - p_1\hat{x_2}(t) > 0, & 0 \le t < t_1, \\ \psi_2(t) - p_1\hat{x_2}(t) \equiv 0, & t_1 \le t \le t_2, \\ \psi_2(t) - p_1\hat{x_2}(t) < 0, & t_2 < t \le \hat{T}. \end{cases} \tag{4.2.1.25}$$

First we assume that $\psi_2(t^*) = p_1\hat{x_2}(t^*)$ for some $0 \le t^* < \hat{T}$. We prove that $\psi_2(t) \le p_1\hat{x_2}(t)$ for all $t^* \le t \le \hat{T}$. On the contrary, we assume that there exist $t'' > t' > t^*$ such that $\psi_2(t) > p_1\hat{x_2}(t)$ for all $t' < t < t''$. By (4.2.1.23),

$$\begin{aligned} \dot{\psi_2} &= p_1\beta - \psi_1 + \psi_2 r'(x_2) \\ \dot{\hat{x_2}} &= \beta - r(\hat{x_2}) \quad \text{on} \quad (t', t''). \end{aligned} \tag{4.2.1.26}$$

Repeating the same argumentation as in the proof of the case (1), it can be shown that $\dot{\hat{x_2}} > 0$ for all $t' < t < t''$. By (4.2.1.26), $\dot{\psi_2}$ is the increasing function of t and $\dot{\hat{x_2}}$ is decreasing function of t on (t', t'') which means that $\psi_2(t) > p_1\hat{x_2}(t)$ for all $t' < t \le \hat{T}$.This is the contradiction with the condition $x_2(\hat{T}) = 0$.

Using this fact we prove(4.2.1.25). Clearly $\psi_2(0) > 0$ and there exist $0 < t_1 < \hat{T}$ such that

$\psi_2(t) > p_1\hat{x_2}(t)$ for all $0 < t < t_1$ and $\psi_2(t_1) = p_1\hat{x_2}(t_1)$. Then, there exist $t_1 \leq t < t_2 < t_2' < \hat{T}$ such that $\psi_2(t) = p_1\hat{x_2}(t)$ for $t_1 \leq t \leq t_2$ and $\psi_2(t) < p_1\hat{x_2(t)}$ for all $t_2 < t < t_2'$. (4.2.1.23), $\hat{u(t)} = -\alpha$ for $t_2 < t < t_2'$ and the functions ψ_2, $\hat{x_2}$ on (t_2, t_2') becomes

$$\begin{aligned} \dot{\psi_2} &= -p_1\alpha - \psi_1 + \psi_2 r'(\hat{x_2}) \\ \dot{\hat{x_2}} &= -\alpha - r(\hat{x_2}) \quad \text{on} \quad (t_2, t_2'). \end{aligned}$$

The function $\dot{\psi_2}$ is decreasing and $\dot{\hat{x_2}}$ is increasing on (t_2, t_2'). This implies that $\psi_2(t) < p_1\hat{x_2(t)}$ for $t_2 < t \leq \hat{T}$ and this proves (4.2.1.25).

Thus, we have the following conclusion.

For the case $0 \leq \eta < 1$: There exist t_1, t_2, t_3, where $0 < t_1 \leq t_2 < t_3 < \hat{T}$, such that

$$\hat{u}(t) = \begin{cases} \beta, & t \in [0, t_1)\,, \\ r(\hat{x_2}), & t \in [t_1, t_2)\,, \\ 0, & t \in [t_2, t_3)\,, \\ -\alpha, & t \in \left[t_3, \hat{T}\right]. \end{cases} \tag{4.2.1.27}$$

For the case $\eta = 1$: There exist t_1, t_2, where $0 < t_1 \leq t_2 < \hat{T}$, such that

$$\hat{u}(t) = \begin{cases} \beta, & t \in [0, t_1)\,, \\ r(\hat{x_2}), & t \in [t_1, t_2)\,, \\ -\alpha, & t \in \left[t_2, \hat{T}\right]. \end{cases} \tag{4.2.1.28}$$

4.2.2 Additional Calculations

From (4.2.1.27), i.e., the case $0 \leq \eta < 1$, we need to determine the unknowns t_1, t_2, t_3, and T. Firstly, we use the terminal conditions $x_1(T) = s$ and, $x_2(T) = 0$, and secondly, we convert them into an effective form involving t_1, t_2, t_3, and T. To do so, we find the form of particular solutions on each of the time intervals as follows: First, we by specify the resistance function as $r(x_2) = kx_2$, $k > 0$ and using this, (4.0.2.6) and (4.0.2.7) can be rewritten as

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= u - kx_2. \end{aligned} \tag{4.2.2.1}$$

1. For $t \in [0, t_1)$ with $u(t) = \beta$, we solve (4.2.2.1) by the separation of variables method. That is,

$$\frac{dx_2}{dt} = \beta - kx_2, \quad \text{this implies} \frac{1}{\beta - kx_2} dx_2 = dt \quad \text{then,} \quad \int \frac{1}{\beta - kx_2} dx_2 = \int dt. \tag{4.2.2.2}$$

Let $u = \beta - kx_2$ then,$du = -kdx_2$, $\frac{-1}{k}du = dx_2$. Then, equation (4.2.2.2) can be written as

$$\frac{-1}{k} \int \frac{1}{u} du = \int dt. \text{ This implies that } \frac{-1}{k} \ln|\beta - kx_2| = t - \frac{1}{k} \ln|C_2|.$$

Hence, the general solution of $x_2(t)$ is

$$x_2(t) = \frac{\beta}{k} - \frac{C_2}{k} e^{-kt}. \tag{4.2.2.3}$$

We substitute equation(4.2.2.3) into equation (4.2.2.1), then we solve for x_1,

$$\frac{dx_1}{dt} = \frac{\beta}{k} - \frac{C_2}{k} e^{-kt}, \quad \text{this implies that} \quad dx_1 = \left(\frac{\beta}{k} - \frac{C_2}{k} e^{-kt} \right) dt.$$

Hence, the general solution of $x_1(t)$ is

$$x_1(t) = \frac{\beta}{k} t + \frac{C_2}{k^2} e^{-kt} + C_1.$$

From initial conditions, we find the values of C_1 and C_2.

$$x_2(0) = \frac{\beta}{k} - \frac{C_2}{k} = 0. \text{ This gives } C_2 = \beta.$$

$$x_1(0) = 0 + \frac{C_2}{k} + C_1 = 0, \quad \text{then} \quad C_1 = -\frac{\beta}{k^2}.$$

Thus, the particular solutions of x_1 and x_2 are respectively

$$x_1(t) = \frac{\beta}{k^2} \left(kt - 1 + e^{-kt} \right), \tag{4.2.2.4}$$

$$x_2(t) = \frac{\beta}{k} \left(1 - e^{-kt} \right). \tag{4.2.2.5}$$

2. For $t \in [t_1, t_2)$ with $u(t) = kx_2$, we solve (4.2.2.1) and obtain

$$\dot{x_1} = x_2$$

$$\dot{x_2} = 0. \text{ this gives } x_2 = C_2. \tag{4.2.2.6}$$

Then, we obtain

$$x_1 = C_2 t + C_1. \tag{4.2.2.7}$$

Using the continuity of optimal trajectories on [0,T], we compare (4.2.2.5) and (4.2.2.6) to get C_2. That is,

$$x_2(t_1) = \frac{\beta}{k}\left(1 - e^{-kt_1}\right) = C_2. \text{ From this } C_2 = \frac{\beta}{k}\left(1 - e^{-kt_1}\right).$$

Similarly, by comparing (4.2.2.4) and (4.2.2.7), we get C_1

$$x_1(t_1) = \frac{\beta}{k}\left(1 - e^{-kt_1}\right) t_1 + C_1, \text{ this implies that } \frac{\beta}{k^2}\left(kt_1 - 1 + e^{-kt_1}\right) = \frac{\beta}{k}\left(1 - e^{-kt_1}\right) t_1 + C_1$$

$$C_1 = \frac{\beta}{k}\left(\frac{-1}{k} + \frac{1}{k}e^{-kt_1} + t_1 e^{-kt_1}\right).$$

The particular solutions of x_1 and x_2 are respectively

$$x_1(t) = \frac{\beta}{k}\left(1 - e^{-kt_1}\right) t + \frac{\beta}{k^2}\left(e^{-kt_1} - 1\right) + \frac{\beta}{k}e^{-kt_1} t_1, \tag{4.2.2.8}$$

$$x_2(t) = \frac{\beta}{k}\left(1 - e^{-kt_1}\right). \tag{4.2.2.9}$$

3. For $t \in [t_2, t_3)$ with $u(t) = 0$, we solve (4.2.2.1)

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -kx_2$$

The general solutions of x_2 and x_1 are respectively

$$x_2 = -\frac{C_2}{k}e^{-kt}, \tag{4.2.2.10}$$

$$x_1 = \frac{C_2}{k}e^{-kt} + C_1. \tag{4.2.2.11}$$

Then, we compare (4.2.2.9) and (4.2.2.10), to get C_2,

$$x_2(t_2) = -\frac{C_2}{k}e^{-kt_2}. \text{ This implies } -\frac{C_2}{k}e^{-kt_2} = \frac{\beta}{k}\left(1 - e^{-kt_1}\right) \text{ then, } C_2 = \beta(e^{-kt_1} - 1)e^{kt_2}.$$

Also we compare (4.2.2.8) and (4.2.2.11), to get C_1,

$$x_1(t_2) = -\frac{C_2}{k}e^{-k_2} + C_1. \text{ Then, } \frac{\beta}{k}\left(1 - e^{-kt_1}\right) t_2 + \frac{\beta}{k^2}\left(e^{-kt_1} - 1\right) + \frac{\beta}{k}e^{-kt_1} t_1 = \frac{C_2}{k^2}e^{-kt} + C_1.$$

$$C_1 = \frac{\beta}{k}\left[\left(1 - e^{-kt_1}\right) t_2 + t_1 e^{-kt_1}\right].$$

Hence, the particular solutions of x_1 and x_2 are respectively

$$x_1(t) = \beta\left(\frac{e^{-kt_1} - 1}{k^2}\right) e^{k(t_2 - t)} + \frac{\beta}{k}\left(1 - e^{-kt_1}\right) t_2 + t_1 e^{-kt_1}, \tag{4.2.2.12}$$

$$x_2(t) = -\frac{\beta}{k}\left(e^{-kt_1} - 1\right) e^{k(t_2 - t)}. \tag{4.2.2.13}$$

4. For $t \in [t_3, T]$ with $u(t) = -\alpha$, we solve (4.2.2.1) and obtain

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\alpha - kx_2$$

The general solutions of x_2 and x_1 are respectively

$$x_2(t) = \frac{-\alpha}{k} - \frac{C_2}{k} e^{-kt}, \tag{4.2.2.14}$$

$$x_1(t) = \frac{-\alpha}{k} t + \frac{C_2}{k^2} e^{-kt} + C_1. \tag{4.2.2.15}$$

Then, we compare (4.2.2.12) and (4.2.2.14) to obtain C_2

$x_2(t_3) = \frac{-\alpha}{k} - \frac{C_2}{k} e^{-kt_3} = -\frac{\beta}{k}\left(e^{-kt_1} - 1\right) e^{k(t_2 - t_3)}$. This gives $C_2 = \beta\left(e^{-kt_1} - 1\right) e^{kt_2} - \alpha e^{kt_3}$.

And also we compare (4.2.2.12) and (4.2.2.14), to obtain C_1

$$x_1(t_3) = \frac{-\alpha}{k} t_3 + \frac{C_2}{k^2} e^{-kt_3} + C_1 = \frac{\beta}{k^2}\left(e^{-kt_1} - 1\right) e^{k(t_2 - t_3)} + \frac{\beta}{k}\left[\left(1 - e^{-kt_1}\right) t_2 + t_1 e^{-kt_1}\right]$$

$$C_1 = \frac{\alpha}{k^2} + \frac{\alpha}{k} t_3 + \frac{\beta}{k}\left[\left(1 - e^{-kt_1}\right) t_2 + t_1 e^{-kt_1}\right].$$

The particular solutions of x_1 and x_2 are respectively

$$x_1(t) = -\frac{\alpha}{k} t + \left[\frac{\beta}{k^2}\left(e^{-kt_1} - 1\right) e^{kt_2} - \frac{\alpha}{k^2} e^{kt_3}\right] e^{-kt} + \frac{\beta}{k}\left[\left(1 - e^{-kt_1}\right) t_2 + t_1 e^{-kt_1}\right] + \frac{\alpha}{k}\left(t_3 + \frac{1}{k}\right), \tag{4.2.2.16}$$

$$x_2(t) = -\frac{\alpha}{k} - \left[\frac{\beta}{k}\left(e^{-kt_1} - 1\right) e^{kt_2} - \frac{\alpha}{k} e^{kt_3}\right] e^{-kt}. \tag{4.2.2.17}$$

Using the terminal conditions (4.0.2.9), we can rewrite (4.2.2.17) as

$$x_2(T) = \frac{\beta}{k}\left(1 - e^{-kt_1}\right)e^{k(t_2-T)} + \frac{\alpha}{k}\left(e^{k(t_3-T)} - 1\right) = 0$$

which can be simplified as

$$\beta\left(1 - e^{-kt_1}\right)e^{k(t_2-T)} + \alpha\left(e^{k(t_3-T)} - 1\right) = 0. \tag{4.2.2.18}$$

Similarly, using the terminal conditions (4.0.2.9), we can rewrite (4.2.2.16) as

$$x_1(T) = \frac{\beta}{k^2}\left(e^{-kt_1} - 1\right)e^{k(t_2-T)} + \frac{\beta}{k}\left(t_2 - t_2e^{-kt_1} + t_1e^{-kt_1}\right) + \frac{\alpha}{k^2}\left(1 - e^{k(t_3-T)}\right) + \frac{\alpha}{k}\left(t_3 - T\right) = s$$

and we simplify this condition as

$$\alpha\left(t_3 - T\right) - sk + \beta\left(t_2 - t_2e^{-kt_1} + t_1e^{-kt_1}\right) = 0. \tag{4.2.2.19}$$

It remains to convert the optimal condition into an effective form, that is (4.0.2.10)can be rewritten as

$$\begin{aligned} J = &\int_0^{t_1}\left[p_1\tfrac{\beta^2}{k}\left(1 - e^{-kt}\right) + p_2\right]dt + \int_{t_1}^{t_2}\left[p_1\tfrac{\beta^2}{k}\left(1 - e^{-kt_1}\right)^2 + p_2\right]dt + \int_{t_2}^{t_3}p_2dt \\ &+ \int_{t_3}^{T}\left[-p_1\alpha\eta\left[-\tfrac{\alpha}{k} - \left[\tfrac{\beta}{k}\left(e^{-kt_1} - 1\right)e^{kt_2} - \tfrac{\alpha}{k}e^{kt_3}\right]e^{-kt}\right] + p_2\right]dt \to min. \end{aligned}$$

This gives

$$\begin{aligned} J = &\,p_1\tfrac{\beta^2}{k}\left(t_1 + \tfrac{1}{k}e^{-kt_1} - \tfrac{1}{k}\right) + p_2t_1 + \left[p_1\tfrac{\beta^2}{k}\left(1 - e^{-kt_1}\right)^2 + p_2\right]\left(t_2 - t_1\right) + p_2\left(t_3 - t_2\right) \\ &-p_1\alpha\eta\left[-\tfrac{\alpha}{k}\left(T - t_3\right) + \left[\tfrac{\beta}{k^2}\left(e^{-kt_1} - 1\right)e^{kt_2} - \tfrac{\alpha}{k^2}e^{kt_3}\right]\left(e^{-kT} - e^{-kt_3}\right) + p_2(T - t_3) \to min. \end{aligned}$$

Can be simplified as

$$\begin{aligned} J = &\,p_1\tfrac{\beta^2}{k}\left(t_2 - t_1\right)\left(1 - e^{-kt_1}\right)^2 + p_1\tfrac{\beta^2}{k^2}\left(kt_1 + e^{-kt_1} - 1\right) - p_1\eta\tfrac{\alpha^2}{k}\left(t_3 - T + \tfrac{1}{k}\left(1 - e^{k(t_3-T)}\right)\right) \\ &+p_2T \to min. \end{aligned} \tag{4.2.2.20}$$

Also, we have an inequality constraint

$$0 < t_1 \leq t_2 < t_3 < T. \tag{4.2.2.21}$$

The problem (4.2.2.18)-(4.2.2.21) is a problem of non-linear programming. We wish to minimize the function $J(t_1, t_2, t_3, T)$ (i.e., (4.2.2.20)) of the four variables subject to equality constraints (4.2.2.18), (4.2.2.19) and inequality constrain (4.2.2.21).

Similarly, from (4.2.1.28), i.e., $\eta = 1$, we need to determine the unknowns t_1, t_2, and T. On the intervals $t \in [0,\ t_1)$ and $t \in [t_1, t_2)$ are the same as above. The only difference is on $t \in [t_2, T)$ and for this, we solve (4.2.2.1) with $u(t) = -\alpha$, we obtain the particular solutions of $x_2(t)$ and $x_1(t)$ respectively

$$x_2(t) = -\frac{\alpha}{k} + \frac{\alpha}{k} e^{k(t_2-t)} + \frac{\beta}{k}\left(1 - e^{-kt_1}\right) e^{k(t_2-t)} \tag{4.2.2.22}$$

$$x_1(t) = -\frac{\alpha}{k} t - \frac{1}{k^2}\left[\alpha e^{kt_2} + \beta\left(1 - e^{-kt_1}\right) e^{kt_2}\right] e^{-kt} + \frac{\beta}{k}\left[\left(1 - e^{-kt_1}\right) t_2 + e^{-kt_1} t_1\right] + \frac{\alpha}{k}\left(t_2 + \frac{1}{k}\right) \tag{4.2.2.23}$$

Using the terminal conditions (4.0.2.9), the conditions (4.2.2.22)and (4.2.2.23) can be rewritten respectively

$$-\frac{\alpha}{k} + \frac{\alpha}{k} e^{k(t_2-T)} + \frac{\beta}{k}\left(1 - e^{-kt_1}\right) e^{k(t_2-T)} = 0, \tag{4.2.2.24}$$

$$-\frac{\alpha}{k} T - \frac{1}{k^2}\left[\alpha e^{kt_2} + \beta\left(1 - e^{-kt_1}\right) e^{kt_2}\right] e^{-kT} + \frac{\beta}{k}\left[\left(1 - e^{-kt_1}\right) t_2 + e^{-kt_1} t_1\right] + \frac{\alpha}{k}\left(t_2 + \frac{1}{k}\right) = s. \tag{4.2.2.25}$$

The conditions (4.2.2.24)and (4.2.2.25) can be simplified respectively

$$\beta\left(1 - e^{-kt_1}\right) e^{k(t_2-T)} + \alpha\left(e^{k(t_2-T)} - 1\right), = 0 \tag{4.2.2.26}$$

$$\alpha\left(t_2 - T\right) - sk + \beta\left(t_2 - t_2 e^{-kt_1} + t_1 e^{-kt_1}\right) = 0. \tag{4.2.2.27}$$

Finally, (4.0.2.10) rewritten as

$$J = p_1 \frac{\beta^2}{k}\left(t_2 - t_1\right)\left(1 - e^{-kt_1}\right)^2 + p_1 \frac{\beta^2}{k^2}\left(kt_1 + e^{-kt_1} - 1\right) - p_1 \eta \frac{\alpha^2}{k}\left(t_2 - T + \frac{1}{k}\left(1 - e^{k(t_2-T)}\right)\right)$$

$$+ \mathrm{p}_2 T \to min. \tag{4.2.2.28}$$

$$0 < t_1 \leq t_2 < T. \tag{4.2.2.29}$$

So, the problem (4.2.2.26)-(4.2.2.29) is a non-linear programming for the case $\eta = 1$. We wish to minimize the function $J(t_1, t_2, T)$ (i.e., (4.2.2.28)) of the three variables subject to equality constraints (4.2.2.26), (4.2.2.27) and inequality constrain (4.2.2.29).

Remark 6: The non-linear programmings (4.2.2.18)-(4.2.2.21) and (4.2.2.26)-(4.2.2.29) are classified as:

1. The minimum-time problem when $p_1 = 0$, $p_2 = 1$. The main question is to find the value of T_{min} such that (4.1.1.7) holds.

2. The minimum-energy problem, when $p_1 = 1, p_2 = 0$. For this problem, T is a prescribed value and α, β,l and k are fixed.

   The main question is to find (numerically) T_{cr} such that $t_1 = t_2$ for $T \leq T_{cr}$ and $t_1 < t_2$ for $T > T_{cr}$.

3. The minimum-time-energy problem when $0 < p_1 < 1$ and $0 < p_2 < 1$, $p_1 + p_2 = 1$. For this case, T is unknown value(along with t_1, t_2, and t_3).

   The main question is to find (numerically) p_{cr} such that $t_1 = t_2$ for $p_1 \leq p_{cr}$ and $t_1 < t_2$ for $p_1 > p_{cr}$.

4. The numerical solutions of these problems are given in the next sections.

5 Numerical Experiments

5.1 The Minimum-Time Problem

The minimum-time Time problem is the case when $p_1 = 0$ and $p_2 = 1$ from (4.2.2.20). For the fixed parameters (α, β, η, k, s), we can rewrite the non-linear programming (4.2.2.18)-(4.2.2.21). Hence, using especial values of $\alpha = \beta = k = s = 1$, we deal with minimum-time

$$T \to min \tag{5.1.0.30}$$

$$\left(1 - e^{-t_1}\right) e^{t_2 - T} + e^{t_3 - T} - 1, = 0 \tag{5.1.0.31}$$

$$t_3 - T - 1 + t_2 - t_2 e^{-t_1} + t_1 e^{-t_1} = 0, \tag{5.1.0.32}$$

$$0 < t_1 \leq t_2 < t_3 < T. \tag{5.1.0.33}$$

This non-linear programming problem has been solved using the software GAMS(General Algebraic Modelling System). We have chosen nlp(non-linear programming) solver.
As we change the values of the fixed parameters for numerical experiment, we must modify (5.1.0.30)-(5.1.0.33) accordingly.
The summary of this numerical solutions (timetabling) given by the following Table 1.

Table 1: Numerical Solutions of Minimum-Time Problem

k	$t_1 = t_2 = t_3 = t_s$	$T = T_{min}$
0.5	1.271	2.042
1	1.585	2.170
1.5	1.944	2.388
2	2.344	2.689
2.5	2.777	3.054

α=β=s=1

Comments:

- The value of T_{min} has a key role in the following section (the explanation in detail will be given later).

- As k (the coefficient of the resistance function) increases, T_{min} increases. This is indeed practical.

- For this non-linear programming, the value of η can be any constant $(0 \le \eta \le 1)$ since it multiply with $p_1 = 0$.

- T_{min} can also be determined analytically as follows.

From (4.1.1.7), we find t_s and T_{min} on $0 \le t < t_s$ and $t_s \le t < T$. We solve the system (4.2.2.1) with $u(t) = \beta$ on $0 \le t < t_1$ and we have already solved, the solution is given by (4.2.2.4) and (4.2.2.5). On the $t_s \le t < T_{min}$, we solve the system (4.2.2.1) with $u(t) = -\alpha$ and the general solutions are given by (4.2.2.14) and (4.2.2.15). By using the continuity of optimal trajectory on $[0, T_{min}]$, we obtain the particular solutions of $x_1(t)$ and $x_2(t)$ respectively

$$x_1(t) = -\frac{\alpha}{k}t + \left(\frac{\alpha}{k^2}e^{kt_s} + \frac{\beta}{k^2}\left(e^{kt_s} - 1\right)e^{-kt} + \frac{\alpha}{k^2} + \frac{\beta}{k}t_s + \frac{\alpha}{k}t_s, \quad (5$$

$$x_2(t) = -\frac{\alpha}{k} + \left(\frac{\alpha}{k}e^{kt_s} + \frac{\beta}{k}\left(e^{kt_s} - 1\right)\right)e^{-kt}. \tag{5.1.0.35}$$

Using terminal conditions $x_1(T) = s$ and $x_2(T) = 0$, we obtain two equations with unknown t_s and T_{min}.

$$-\frac{\alpha}{k}T_{min} + \left(\frac{\alpha}{k^2}e^{kt_s} + \frac{\beta}{k^2}\left(e^{kt_s} - 1\right)\right)e^{-kT_{min}} + \frac{\alpha}{k^2} + \frac{\beta}{k}t_s + \frac{\alpha}{k}t_s = s \tag{5.1.0.36}$$

$$-\frac{\alpha}{k} + \left(\frac{\alpha}{k}e^{kt_s} + \frac{\beta}{k}\left(e^{kt_s} - 1\right)\right)e^{-kT_{min}} = 0. \tag{5.1.0.37}$$

From (5.1.0.37), we solve for t_s and substitute into (5.1.0.36) gives non-linear equation with unknown T_{min}

$$\begin{aligned} &\tfrac{\beta}{k^2}\left(\tfrac{\alpha+\beta}{\alpha e^{kT_{min}}+\beta} - 1\right) + \tfrac{\beta}{k}\ln\left(\tfrac{\beta+\alpha e^{kT_{min}}}{\alpha+\beta}\right) + \tfrac{\alpha}{k^2}\left(e^{kT_{min}}\tfrac{\alpha+\beta}{\alpha e^{kT_{min}}+\beta} - 1\right) \\ &+\tfrac{\alpha}{k}\left(\ln\tfrac{\beta+\alpha e^{kT_{min}}}{\alpha+\beta} - T_{min}\right) - s = 0 \end{aligned} \tag{5.1.0.38}$$

We can see that from (5.1.0.38), T_{min}, it is a logarithmic function. It is solvable when the domain is greater than zero.

5.2 The Minimum-Energy Problem

First, we discuss the minimum energy problem (i.e., $p_1 = 1$ and $p_2 = 0$) for the case $0 \leq \eta < 1$ where T is a prescribed value. Then, the main problem is to find (numerically) T_{cr} such that $t_1 = t_2$ for $T \leq T_{cr}$ and $t_1 < t_2$ for $T > T_{cr}$ (α, β, k, and s are fixed). Second, we discuss this problem for $\eta = 1$.

Now, for the first case, we discuss the numerical solutions by taking the special values $\alpha = \beta = s = k = 1$, $\eta = 0$. Using these special values, one can determine the values of the T_{min} from (5.1.0.38). In our case, we have already solved the minimum-time energy. Therefore, $T_{min} = 2.170$ (see, Table 1). Then the non-linear programming problem (4.2.2.18)-(4.2.2.21) becomes minimum-energy problem and given as follows:

$$J = (t_2 - t_1)\left(1 - e^{-t_1}\right)^2 + t_1 + e^{-t_1} - 1 \rightarrow min \tag{5.2.0.39}$$

$$\left(1 - e^{-t_1}\right) e^{t_2 - T} + e^{t_3 - T} - 1 = 0 \tag{5.2.0.40}$$

$$t_3 - T - 1 + t_2 - t_2 e^{-t_1} + t_1 e^{-t_1} = 0. \tag{5.2.0.41}$$

$$0 < t_1 \leq t_2 < t_3 < T \tag{5.2.0.42}$$

The numerical solutions show that the timetabling of the train as: there exists the critical values T_{cr} such that if $T_{min} < T \leq T_{cr}$, the non-linear programming (5.2.0.39)-(5.2.0.42) has the optimal solution $t_1 = t_2 < t_3$ (the optimal regime with resistance function does not occur). If $T > T_{cr}$, the non-linear programming (5.2.0.39)-(5.2.0.42) has the optimal solution such that $t_1 < t_2 < t_3$ (all four optimal regimes occur). In the same way, we can discuss this problem for different values of k. The summary of this numerical solutions (timetabling) given by the following Table 2.

Table 2: Numerical Solutions of Minimum-Energy for the Case $0 \leq \eta < 1$

k	$T = T_{min}$	t_1=t_2=t_3	T=T_{cr}	t_1=t_2	t_3	$T > T_{cr}$	t_1	t_2	t$_3$
0.5	2.042	1.271	2.490	0.811	2.179	6.000	0.192	4.524	5.910
1	2.170	1.585	2.315	1.312	2.003	6.000	0.195	5.222	5.915
1.5	2.388	1.944	2.470	1.754	2.216	6.000	0.203	5.456	5.918
2	2.689	2.344	2.740	2.209	2.531	6.000	0.214	5.573	5.920

α=β=s=p_1=1 and η= $p_2 = 0$

For the second case ($\eta = 1$), by taking the special values α=β=s=p_1=k=1 and $p_2 = 0$, the non-linear programming (4.2.2.26)-(4.2.2.26) be becomes energy optimal problem and given as follows:

$$J = (t_2 - t_1)\left(1 - e^{-t_1}\right)^2 + t_1 + e^{-t_1} + T - T_3 - e^{T-t_1} \rightarrow min \tag{5.2.0.43}$$

$$\left(1 - e^{-t_1}\right) e^{t_2 - T} + e^{t_3 - T} - 1 = 0 \tag{5.2.0.44}$$

$$t_3 - T - 1 + t_2 - t_2 e^{-t_1} + t_1 e^{-t_1} = 0 \tag{5.2.0.45}$$

$$0 < t_1 \leq t_2 < T. \tag{5.2.0.46}$$

The summary of this numerical solutions (timetabling) given by the following Table 3.

Table 3: Numerical Solutions of Minimum-Energy for the Case $\eta = 1$

k	$T = T_{min}$	t_1=t_2	$T > T_{min}$	t_1	t_2
0.5	2.042	1.271	6.000	0.179	5.835
1	2.170	1.585	6.000	0.188	5.842
1.5	2.388	1.944	6.000	0.198	5.847
2	2.689	2.344	6.000	0.210	5.852

α=β=s=p_1=1 and $p_2 = 0$

Comments:

- We can see that from Table 2, T_{cr} is very important to identify optimal driving regimes.

- As k (the coefficient of the resistance function) increases, T_{min} increases. This is indeed practical.

5.3 The Minimum-Time-Energy Problem

The minimum-time-energy problem is the case where $0 < p_1 < 1$ and $0 < p_2 < 1$ such that $p_1 + p_2 = 1$. First, we discuss the numerical solutions for the case $0 \leq \eta < 1$ by taking special values $\alpha = \beta = s = k = 1$, $\eta = 0$. Then, we find (numerically) the value of P_{cr}. Second, we discuss this problem for $\eta = 1$.
Hence, for the first case, we deal with minimum time-energy problem

$$J = p_1 (t_2 - t_1) \left(1 - e^{-t_1}\right)^2 + p_1 \left(t_1 + e^{-t_1} - 1\right) + p_2 T \rightarrow min \tag{5.3.0.47}$$

Subject to equality constraints (5.2.0.44),(5.2.0.45), and inequality constraint (5.2.0.46). As we change the values of the fixed parameters for numerical experiment, we must modify (5.3.0.47), (5.2.0.44),(5.2.0.45) and (5.2.0.46) accordingly. The summary of this numerical solutions (timetabling) given by the following Table 4.

Table 4: Numerical Solutions of Minimum-Time-Energy for the Case $0 \leq \eta < 1$

η	p_{cr}	t_1=t_2	t_3	T	$p_1 > p_{cr}$	p_2	t_1	t_2	t_3	T
0	0.649	1.313	2.000	2.313	0.660	0.340	1.265	1.323	2.016	2.323
0.1	0.649	1.320	1.984	2.304	0.660	0.340	1.265	1.333	2.000	2.313
0.5	0.640	1.364	1.892	2.256	0.650	0.350	1.324	1.370	1.905	2.262

α=β=s=k=1

Table 5: Numerical solutions of Minimum-Time-Energy for the Case $\eta = 0.5$

k	p_{cr}	t_1=t_2	t_3	T	$p_1 > p_{cr}$	p_2	t_1	t_2	t_3	T
0.1	0.770	0.718	1.550	2.168	0.950	0.05	0.459	0.461	2.505	2.164
0.5	0.650	1.008	1.637	2.144	0.800	0.200	0.873	0.884	1.953	2.330

α=β=s=1 and $\eta = 0.5$

Table 6: Numerical Solutions of Minimum-Time-Energy for the Case $\eta = 1$

k	p_{cr}	t_1=t_2	T	$p_1 > p_{cr}$	p_2	t_1	t_2	T
0.1	0.00003	1.051	2.002	0.001	0.999	1.040	1.060	2.002
0.5	0.0001	1.271	2.042	0.01	0.99	1.244	1.281	2.042
1	0.0003	1.585	2.170	0.03	0.97	1.541	1.590	2.170
1.5	0.0002	1.944	2.388	0.02	0.98	1.927	1.944	2.388
2	0.001	2.344	2.689	0.1	0.9	2.286	2.345	2.689

α=β=s=1

Comments:

- We can see that from Table 2, T_{cr} is very important to identify optimal driving regimes.
- From Table 4, as η increases, T_{min} decreases. This means that energy is saved.
- From Table 6, for those particular parameters, we see that T is minimum-time (see, Table 1).

6 Conclusions

The dynamics of an electric train journey is usually studied with respect to a search for its energy-efficient control. This problem can be solved in the frame of different variants by use of suitable methods of continuous and discrete optimization. This thesis set out to address the following objectives: First, to review a survey of basic mathematical models used in energy-efficient train controls. Second, to apply of Pontryagin's Maximum Principle to a given model. Third, to analysis of optimal driving regimes. Fourth, to discuss timetabling of energy-efficient train control.

We formulated the basic energy-efficient train control model by using Newton's second law of motion and other known physical laws on the level track under assumption of standard resistance function, as well as some generalizations of the problem from mathematical point of view. By applying Pontryagin's Maximum Principle to the formulated model, the problem is mainly categorized into minimum-time, minimum-energy, and minimum-time-energy problem depending on the solution of adjoint system ($\psi_0 = C_0 \leq 0$) from Maximum Principle and corresponding to the weight parameters. For the minimum-time problem, we derived two optimal driving regime and we proved that the switching time between the optimal driving regimes occurred according to the sequence of optimal driving regimes. For the minimum time-energy problem, by considering the special parameter $0 \leq \eta \leq 1$ which represents the portion of the electric energy that is being reloaded to the electric circuit (grid) while braking, we discussed two cases. For the case $0 \leq \eta < 1$, we found four optimal driving regimes and we proved that the switching times between these optimal driving regimes occurred according to the sequence of the optimal driving regimes. And for the case $\eta = 1$, we found three optimal driving regimes and also we proved the switching time between these optimal driving regimes occurred according to the sequence of these optimal driving regimes.

Finally, we settled the Non-linear programming. The numerical results of this non-linear programming show that for minimum-energy problem for the case $0 \leq \eta < 1$, there exist a critical parameter T_{cr} and T_{min} with following timetabling properties: If $T < T_{min}$, the system has no solution. If $T = T_{min}$ the system has a solution ($t_1 = t_2 = t_3$). If

$T_{min} < T \leq T_{cr}$, then, system has the solution $t_1 = t_2 < t_3$ (i.e., driving with resistance function does not occur). If $T > T_{cr}$, the system has the solution $t_1 < t_2 < t_3$ (i.e., all four optimal driving regimes occur). For the case $\eta = 1$, there exists a T_{min} with the following timetabling properties: If $T < T_{min}$, the system has no solution. If $T = T_{min}$, the system has the solution $t_1 = t_2$ (i.e., the driving with resistance function does not occurred). If $T_{min} < T$ has the solution $t_1 < t_2$ (i.e., all the three optimal driving regimes occur).

For minimum time-energy problem, we considered the weight parameters $0 < p_1 < 1$, $0 < p_2 < 1$ such that $p_1 + p_2 = 1$ (since we are dealt with combination of minimum-time, minimu-energy, and minimum-time-energy). The numerical results show that for the case $0 \leq \eta < 1$ there exists a critical parameter p_{cr} the following timetabling properties: If $p_1 \leq p_{cr}$, the system has the solution $t_1 = t_2 < t_3$ (i.e., the driving with resistance does not occur). If $p_1 > p_{cr}$, the system has the solution $t_1 < t_2 < t_3$ (i.e, all four optimal driving regimes occur). For the case $\eta = 1$ there exists a critical parameter p_{cr} the following timetabling properties: If $p_1 \leq p_{cr}$, the system has the solution $t_1 = t_2$ (i.e., the driving with resistance does not occur). If $p_1 > p_{cr}$, the system has the solution $t_1 < t_2$ (i.e, all three driving optimal regimes occur).

In this thesis, we have solved the problem of energy-efficient train control by considering an electric consumption model on the level track. By its nature, it is a continuous model. However, we can consider this model as a discrete one. Another possible research for this problem is to consider an electric consumption model on non-level track. And also, fuel consumption model by considering on level and non-level track.

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8 Appendix

GAMS Code for Minimum-Time Problem for $0 \leq \eta < 1$ (see Table 1)

```
Option nlp=ipopth;

Option ResLim = 100000;

$ontext

First option refers to chosen solver for Non-Linear Programming. More about

solvers can be found on https://www.gams.com/latest/docs/S_MAIN.html

Second option changes computation time limit (in seconds). Default setting

is 1000 seconds.

$offtext


Scalars

a    /1/

p1   /0/

p2   /1/

b    /1/

e    /0/

k    /1/

s    /1/;


$ontext

 We have introduced the value of the scalars (see Table 1).

"a" = alpha, "b" = beta, "e" = eta.

Next we introduce "variables" which are defined default as free variables.

Then the easiest way is define them once more as "positive"/"negative"/"binary"

etc. as shown below.

$offtext


Variables
```

```
t1
t2
t3
T
z        objective variable;

Positive variables
t1
t2
t3
T;

$ontext
Then you define name of equations you're going to use. You need to define all
of the names!
$offtext

Equations  obj     objective function
           const1
           const2
           c1
           c2
           c3;
$ontext
Next step is to define equations. You need to use =l/e/g=, which means
less or equal / equal / greater or equal. GAMS is not able to make strict
inequalities. If you need them, the only way is to use some small epsilon
scalar. Be careful with brackets. Equations are easy to understand, here
for example (b**2) means b powered by 2, but you can also use b*b.
$offtext
```

```
obj..    z =e= p1*((b**2)/k)*(t2-t1)*(1-exp(-k*t1))**2 +
               p1*((b**2)/(k**2))*(k*t1 + exp(-k*t1) - 1) -
               p1*e*((a**2)/k)*(t3 - T + (1/k)*(exp(k*(T-t3))-1)) + p2*T;
const1.. 0 =e= b*(1-exp(-k*t1))*exp(k*(t2-T)) + a*(exp(k*(t3-T)) - 1);
const2.. 0 =e= a*(t3-T) - s*k + b*(t2-t2*exp(-k*t1) + t1*exp(-k*t1));
c1..     t1 =l= t2;
c2..     t2 =l= t3;
c3..     t3 =l= T;
$ontext
Last step is to define models. First we need to write
Model "model name" /constraints used in model/;
Solve "model name" minimizing/maximizing "name of objective variable" using nlp
Display - here we put parameters/variables we want to see as the result. In case
we want to show variables, we need to write "variable name".l
$offtext

Model Berkessa /obj, const1, const2, c1, c2, c3/;
Solve Berkessa minimizing z using nlp;
Display t1.l, t2.l, t3.l, T.l, z.l;
```

GAMS Code for Minimum-energy Problem for $0 \leq \eta < 1$ (see Table 2)

```
Option nlp=ipopth;
Option ResLim = 100000;

Scalars
a    /1/
p1   /1/
p2   /0/
b    /1/
```

```
e    /0/
k    /1/
s    /1/;

Variables
t1
t2
t3
T
z        objective variable;

Positive variables
t1
t2
t3
T;
Equations  obj     objective function
           const1
           const2
           c1
           c2
           c3
           c5
           c6
           c7;
obj..    z =e= p1*((b**2)/k)*(t2-t1)*(1-exp(-k*t1))**2 +
               p1*((b**2)/(k**2))*(k*t1 + exp(-k*t1) - 1) -
               p1*e*((a**2)/k)*(t3 - T + (1/k)*(exp(k*(T-t3))-1)) + p2*T;
const1.. 0 =e= b*(1-exp(-k*t1))*exp(k*(t2-T)) + a*(exp(k*(t3-T)) - 1);
const2.. 0 =e= a*(t3-T) - s*k + b*(t2-t2*exp(-k*t1) + t1*exp(-k*t1));
```

```
c1..     t1 =l= t2;
c2..     t2 =l= t3;
c3..     t3 =l= T;
c5..     T =e= 2.042;
c6..     T =e= 2.170;
c7..     T =e= 6.000;
Model Hube1 /obj, const1, const2, c1, c2, c3,c7/;
Solve Hube1 minimizing z using nlp;
Display t1.l, t2.l, t3.l, T.l, z.l;
Model Hube2 /obj, const1, const2, c1, c2, c3, c6/;
Solve Hube2 minimizing z using nlp;
Display t1.l, t2.l, t3.l, T.l, z.l;
Model Hube3 /obj, const1, const2, c1, c2, c3, c5/;
Solve Hube3 minimizing z using nlp;
Display t1.l, t2.l, t3.l, T.l, z.l;
```

GAMS Code for Minimum-energy Problem for $\eta = 1$ (see Table 3)

```
Option nlp=ipopth;
Option ResLim = 100000;

Scalars
a    /1/
p1   /1/
p2   /0/
b    /1/
e    /1/
k    /0.1/
s    /1/;

Variables
```

```
t1
t2
T
z        objective variable;

Positive variables
t1
t2
T;
Equations  obj     objective function
           const1
           const2
           c1
           c2
           c3
           c5
           c6
           c7;
obj..    z =e= p1*((b**2)/k)*(t2-t1)*(1-exp(-k*t1))**2 +
               p1*((b**2)/(k**2))*(k*t1 + exp(-k*t1) - 1) -
               p1*e*((a**2)/k)*(t2 - T + (1/k)*(1-exp(-k*(t2-T)))) + p2*T;
const1.. 0 =e= b*(1-exp(-k*t1))*exp(k*(t2-T)) + a*(exp(k*(t2-T)) - 1);
const2.. 0 =e= a*(t2-T) - s*k + b*(t2-exp(-k*t1)*t2 + t1*exp(-k*t1));

c1..     t1 =l= t2;
c2..     t2 =l= T;
c5..     T =e= 2.042;
c6..     T =e= 2.170;
c7..     T =e= 6.000;
Model Boru1 /obj, const1, const2, c2/;
```

```
Solve Boru1 minimizing z using nlp;
Display t1.l, t2.l, T.l, z.l;
 Model Boru2 /obj, const1, const2, c2/;
Solve Boru2 minimizing z using nlp;
Display t1.l, t2.l, T.l, z.l;

Model Boru3 /obj, const1, const2, c1, c2/;
Solve Boru3 minimizing z using nlp;
Display t1.l, t2.l, T.l, z.l;
```

GAMS Code for Minimum-Time-Energy Problem for $0 \leq \eta < 1$ (see Table 4 and Table 5)

```
Option nlp=ipopth;
Option ResLim = 100000;

Scalars
a    /1/
p1   /0.649/
p2   /0.351/
b    /1/
e    /1/
k    /0.1/
s    /1/;
$ontext
In scalars you introduce constant values.
"a" = alpha, "b" = beta, "e" = eta.
$offtext

Variables
t1
```

```
t2
t3
T
z        objective variable;

Positive variables
t1
t2
t3
T;
Equations  obj     objective function
           const1
           const2
           c1
           c2
           c3;
obj..    z =e= p1*((b**2)/k)*(t2-t1)*(1-exp(-k*t1))**2 +
               p1*((b**2)/(k**2))*(k*t1 + exp(-k*t1) - 1) -
               p1*e*((a**2)/k)*(t3 - T + (1/k)*(exp(k*(T-t3))-1)) + p2*T;
const1.. 0 =e= b*(1-exp(-k*t1))*exp(k*(t2-T)) + a*(exp(k*(t3-T)) - 1);
const2.. 0 =e= a*(t3-T) - s*k + b*(t2-t2*exp(-k*t1) + t1*exp(-k*t1));
c1..     t1 =l= t2;
c2..     t2 =l= t3;
c3..     t3 =l= T;

Model Zew1 /obj, const1, const2, c1, c2, c3/;
Solve Zew1 minimizing z using nlp;
Display t1.l, t2.l, t3.l, T.l, z.l;
```

GAMS Code for Minimum-Time-Energy Problem for $\eta = 1$ (see Table 6)

```
Option nlp=ipopth;
Option ResLim = 100000;
Scalars
a    /1/
p1   /0.001/
p2   /0.999/
b    /1/
e    /1/
k    /0.1/
s    /1/;

Variables
t1
t2
T
z        objective variable;

Positive variables
t1
t2
T;
Equations  obj     objective function
           const1
           const2
           c1
           c2;
obj..    z =e= p1*((b**2)/k)*(t2-t1)*(1-exp(-k*t1))**2 +
               p1*((b**2)/(k**2))*(k*t1 + exp(-k*t1) - 1) -
               p1*e*((a**2)/k)*(t2 - T + (1/k)*(1-exp(-k*(t2-T)))) + p2*T;
const1.. 0 =e= b*(1-exp(-k*t1))*exp(k*(t2-T)) + a*(exp(k*(t2-T)) - 1);
```

```
const2.. 0 =e= a*(t2-T) - s*k + b*(t2-exp(-k*t1)*t2 + t1*exp(-k*t1));
c1..     t1 =l= t2;
c2..     t2 =l= T;

Model Zewude /obj, const1, const2, c2/;
Solve Zewude minimizing z using nlp;
Display t1.l, t2.l, T.l, z.l;
```